

A UNIFORM ASYMPTOTIC FORMULA FOR THE SECOND MOMENT OF PRIMITIVE L -FUNCTIONS ON THE CRITICAL LINE

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ABSTRACT. We prove an asymptotic formula for the second moment of automorphic L -functions of even weight and prime power level. The error term is estimated uniformly in all parameters: level, weight, shift and twist.

CONTENTS

1. Introduction	1
2. Preliminary information	4
3. Primitive forms and the Petersson trace formula	5
4. Integrals involving the Bessel functions	7
4.1. Proof of theorem 4.4	9
5. Hypergeometric functions	14
6. The second moment: $\nu \geq 3$	18
7. The second moment: $\nu = 2$	25
7.1. The sum S_4	28
7.2. The sum S_3	31
7.3. The sum S_2	35
8. Error terms	38
Funding	44
References	44

1. INTRODUCTION

Many problems of different nature can be phrased in terms of central L -values. Therefore, asymptotic evaluation of moments of L -functions near the critical line is a major subject in analytic number theory. In view of applications, one of the most important things is to estimate

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error terms optimally and uniformly with respect to different parameters.

In this paper, we consider the family of automorphic L -functions associated to primitive forms of weight $2k \geq 2$ and level $N = p^\nu$ with p prime and $\nu \geq 2$.

Let γ be the Euler constant, $\Lambda(n)$ be the von Mangoldt function, $\phi(n)$ be the Euler-totient function, $\mu(n)$ be the Möbius function and $\psi(n)$ be the logarithmic derivative of the Gamma-function. For complex s and natural n define

$$(1.1) \quad \tau_s(n) = \sum_{n_1 n_2 = n} \left(\frac{n_1}{n_2} \right)^s = n^{-s} \sigma_{2s}(n).$$

Introduce the identity operator

$$(1.2) \quad \mathbf{1}_c = \begin{cases} 1 & \text{if } c \text{ is true} \\ 0 & \text{otherwise} \end{cases}.$$

Let p be a prime and

$$(1.3) \quad \phi_\nu(N) = \begin{cases} 1 - p^{-1}, & \text{if } N = p^\nu, \nu \geq 3; \\ 1 - (p - p^{-1})^{-1}, & \text{if } N = p^\nu, \nu = 2. \end{cases}$$

Our main result is the asymptotic formula for the twisted harmonic second moment $M_2(l, 0, it)$ defined by (3.11).

Theorem 1.1. *Let $k \geq 1$, $t \in \mathbf{R}$, $T := 3 + |t|$, $N = p^\nu$, p is a prime and $\nu \geq 2$. If $p|l$ then $M_2(l, 0, it) = 0$. Otherwise*

$$(1.4) \quad M_2(l, 0, it) = \frac{\phi(N)\phi_\nu(N)}{N} \frac{\sigma_{-2it}(l)}{l^{1/2-it}} \times \\ \times \left(\log N + 2\gamma - 2\log 2\pi + 2\frac{\log p}{p-1} + \psi(k+it) + \psi(k-it) \right) - \\ - \frac{\phi(N)\phi_\nu(N)}{N} \frac{1}{l^{1/2-it}} \sum_{d|l} \sum_{r|d} \mu(d/r) \tau_{it}(r) r^{-it} \log r + \\ + O_\epsilon \left(V_N(l) + \frac{1}{p} V_{N/p}(l) + \frac{1}{p^2} V_{N/p^2}(l) + \frac{(lTkp)^\epsilon}{p^2 \sqrt{l}} \mathbf{1}_{N=p^2} \right).$$

Here

$$(1.5) \quad V_N(l) \ll \begin{cases} (lkT^2)^\epsilon \frac{l^{1/2}(k+T)}{N} & l \geq N/(1+T^2); \\ \left(\frac{lT^2}{4(N-l)} \right)^k \frac{N^\epsilon}{l^{1/2T}} & l < N/(1+T^2). \end{cases}$$

The given asymptotic formula is uniform in all parameters: N , p , l , k and T . Fixing k we obtain that

$$(1.6) \quad V_N(l) + \frac{1}{p}V_{N/p}(l) + \frac{1}{p^2}V_{N/p^2}(l) \ll_{k,\epsilon} \frac{l^{1/2}T}{N}(lNT)^\epsilon.$$

This improves and generalizes several results in the prior literature.

First, the given paper extends methods of [5, 6, 7] to the case of primitive forms of prime power level. We apply the technique of analytic continuation instead of approximate functional equation in order to prove the explicit formula for the twisted second moment on the critical line. This generalizes the formulas of Bykovskii [5], Iwaniec & Sarnak [11] and Bykovskii & Frolenkov [6]. The most challenging case of weight $2k = 2$ is proved in [6] using the so-called "Hecke trick".

From the explicit formula we derive the asymptotics that generalizes theorem 1 of [4] proved for N prime, $k = 1$, $t = 0$ and $l < N$. Removing the restriction $l < N$ is of crucial importance here. As shown in [1] this is required to prove the best known lower bound on the proportion of non-vanishing L -values when $N = p^\nu$, ν is fixed and $p \rightarrow \infty$.

Furthermore, uniformity on shift t allows establishing a positive proportion of non-vanishing at any point on the critical line $1/2 + it$ such that $|t| < N$.

Finally, theorem 1.1 improves the result of Rouymi.

Theorem 1.2. (*lemma 5 of [14]*) *Let $k \geq 1$, $N = p^\nu$, p is prime, $\nu \geq 3$, $(l, p) = 1$. Then for all $1 \leq l \leq N$*

$$(1.7) \quad M_2(l, 0, 0) = \frac{\tau(l)}{\sqrt{l}} \left(\frac{\phi(N)}{N} \right)^2 \times \\ \times \left(\log \left(\frac{N}{4\pi^2 l} \right) + 2 \left(\frac{\log p}{p-1} + \psi(k) + \gamma \right) \right) + O_{k,p} \left(\sqrt{l} \frac{(\log N)^4}{\sqrt{N}} \right).$$

Note that if $t = 0$, the main term of (1.4) coincides with the main term in (1.7). Indeed,

$$(1.8) \quad \sum_{d|l} \sum_{r|d} \mu(d/r) \tau_0(r) \log r = \sum_{d|l} \sum_{r|d} \mu(r) \log \frac{d}{r} \sum_{a|\frac{d}{r}} 1 = \\ = \sum_{d|l} \sum_{a|d} \sum_{r|\frac{d}{a}} \mu(r) \log \frac{d}{r} = \sum_{d|l} \left(\log d + \sum_{k|d} \Lambda(d/k) \right) = \\ = 2 \sum_{d|l} \log d = \sum_{d|l} \left(\log d + \log \frac{l}{d} \right) = \tau_0(l) \log l.$$

One can also compare the error terms in theorems 1.1 and 1.2 by setting $l = 1$. Equation (1.7) gives $O_{k,p}((\log N)^4 N^{-1/2})$ and (1.5) implies that

$$(1.9) \quad V_N(1) + \frac{1}{p} V_{N/p}(1) + \frac{1}{p^2} V_{N/p^2}(1) \ll_{k,t,\epsilon} \frac{N^\epsilon}{p^2(N/p^2)^k} \ll_{k,t,\epsilon,p} N^{-k+\epsilon}.$$

The paper is organized as follows. In sections 2 and 3 we recall some background information. Sections 4 and 5 are devoted to estimating special functions which appear as a part of error terms in the asymptotic formula for the second moment. In section 6 we prove the exact formula for the second moment when $N = p^\nu$, $\nu \geq 3$. The case $N = p^2$ is considered in section 7. Finally, in the last section we derive the asymptotic formula by estimating uniformly in all parameters different types of error terms.

2. PRELIMINARY INFORMATION

Define

$$\Gamma(u, v, \lambda; s) = \frac{\Gamma(\lambda - 1/2 + s/2)}{\Gamma(\lambda + 1/2 - s/2)} \Gamma(1/2 - u + v - s/2) \Gamma(1/2 - u - v - s/2).$$

Using Stirling's formula for $\Re v = 0$ we have

$$(2.1) \quad \Gamma(u, v, \lambda; \sigma + it) \ll \frac{\exp(-\pi|t|/2)}{|t|^{1+2\Re u}}.$$

The Gauss hypergeometric function is defined for $|z| < 1$ by the power series

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}.$$

The classical delta symbol is denoted by

$$(2.2) \quad \delta(a, b) = \begin{cases} 1, & a = b \\ 0, & \text{otherwise} \end{cases}.$$

For $q \in \mathbf{N}$ and $a \in \mathbf{Z}$ let

$$(2.3) \quad \delta_q(a) = \begin{cases} 1, & a \equiv 0 \pmod{q} \\ 0, & \text{otherwise} \end{cases}.$$

The Bessel function

$$(2.4) \quad J_s(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+s)} \left(\frac{z}{2}\right)^{s+2n}$$

satisfies the Mellin-Barnes representation

$$(2.5) \quad J_{2\lambda-1}(y) = \frac{1}{4\pi i} \int_{\Re s = \Delta} \frac{\Gamma(\lambda - 1/2 + s/2)}{\Gamma(\lambda + 1/2 - s/2)} \left(\frac{y}{2}\right)^{-s} ds$$

for $1 - 2\Re\lambda < \Delta < 0$ and positive real y . The Lerch zeta function with parameters $\alpha, \beta \in \mathbf{R}$ is defined by

$$(2.6) \quad \zeta(\alpha, \beta; s) = \sum_{n+\alpha > 0} \frac{\exp(2\pi i n \beta)}{(n + \alpha)^s}$$

for $\Re s > 1$. This is a periodic function on β with a period one; $\zeta(\alpha, \beta; s)$ can be holomorphically continued on the whole complex plane except the point $s = 1$ for $\beta \in \mathbf{Z}$, where it has a simple pole with residue 1.

The Mellin transform of $f : [0, \infty) \rightarrow \mathbb{C}$ is given by

$$(2.7) \quad \hat{f}(s) = \int_0^\infty f(x) x^{s-1} dx.$$

The classical Kloosterman sum

$$(2.8) \quad Kl(m, n; c) = \sum_{\substack{0 \leq x \leq c-1 \\ (x, c)=1}} \exp\left(2\pi i \frac{mx + n\bar{x}}{c}\right), \quad x\bar{x} \equiv 1 \pmod{c}$$

satisfies the Weil bound

$$(2.9) \quad |Kl(m, n; c)| \leq \tau_0(c) \sqrt{(m, n, c)} \sqrt{c}.$$

Lemma 2.1. ([15], lemma A.12) *Let m, n, c be three strictly positive integers and p be a prime number. Suppose $p^2 | c$, $p | m$ and $p \nmid n$. Then $Kl(m, n; c) = 0$.*

3. PRIMITIVE FORMS AND THE PETERSSON TRACE FORMULA

Let $S_{2k}(N)$ be the space of cusp forms of weight $2k \geq 2$ and level N . It is equipped with the Petersson inner product

$$(3.1) \quad \langle f, g \rangle_N := \int_{F_0(N)} f(z) \overline{g(z)} y^k \frac{dx dy}{y^2},$$

where $F_0(N)$ is a fundamental domain of the action of the Hecke congruence subgroup $\Gamma_0(N)$ on the upper-half plane $\mathbb{H} = \{z \in \mathbf{C} : \Im z > 0\}$.

Any $f \in S_{2k}(N)$ has a Fourier expansion at infinity

$$(3.2) \quad f(z) = \sum_{n \geq 1} a_f(n) e(nz).$$

According to the Atkin-Lehner theory the space of cusp forms can be decomposed into two subspaces of new and old forms

$$(3.3) \quad S_{2k}(N) = S_{2k}^{new}(N) \oplus S_{2k}^{old}(N).$$

We denote by $H_{2k}(N)$ an orthogonal basis of $S_{2k}(N)$ and by $H_{2k}^*(N)$ an orthogonal basis of $S_{2k}^{new}(N)$ consisting of primitive forms with normalized Fourier coefficients

$$(3.4) \quad \lambda_f(n) := a_f(n)n^{-(2k-1)/2},$$

$$(3.5) \quad \lambda_f(1) = 1.$$

The coefficients $\lambda_f(n)$ satisfy the Hecke relation

$$(3.6) \quad \lambda_f(m)\lambda_f(n) = \sum_{\substack{d|(m,n) \\ (d,p)=1}} \lambda_f\left(\frac{mn}{d^2}\right).$$

Let $Re(s) > 1$, then for $f \in H_{2k}^*(N)$ we define an automorphic L -function

$$(3.7) \quad L(s, f) = \sum_{n \geq 1} \lambda_f(n)n^{-s}.$$

The completed L -function

$$(3.8) \quad \Lambda(s, f) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma\left(s + \frac{2k-1}{2}\right) L(s, f)$$

can be analytically continued on the whole complex plane and satisfies the functional equation

$$(3.9) \quad \Lambda(s, f) = \epsilon_f \Lambda(1-s, f),$$

where $s \in \mathbf{C}$ and $\epsilon_f = \pm 1$.

We define the harmonic average as follows

$$(3.10) \quad \sum_{f \in H_{2k}(N)}^h := \sum_{f \in H_{2k}(N)} \frac{\Gamma(2k-1)}{(4\pi)^{2k-1} \langle f, f \rangle_N}.$$

In this paper we study the twisted harmonic second moment of automorphic L -functions associated to primitive forms of weight $2k \geq 2$ and level $N = p^\nu$ with p prime and $\nu \geq 2$

$$(3.11) \quad M_2(l, u, v) = \sum_{f \in H_{2k}^*(N)}^h \lambda_f(l) L_f(1/2 + u + v) L_f(1/2 + u - v).$$

Our proof is based on the Petersson trace formula.

Theorem 3.1. *For $m, n \geq 1$ we have*

$$(3.12) \quad \Delta_{2k,N}(m, n) = \sum_{f \in H_{2k}(N)}^h \lambda_f(m) \lambda_f(n) = \\ = \delta(m, n) + 2\pi i^{-2k} \sum_{N|c} \frac{Kl(m, n; c)}{c} J_{2k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

More precisely, we apply the following generalization by Rouymi.

Theorem 3.2. *(remark 4 of [13]) Let $N = p^\nu$ with prime p and $\nu \geq 2$. Then*

$$(3.13) \quad \Delta_{2k,N}^*(m, n) := \sum_{f \in H_{2k}^*(N)}^h \lambda_f(m) \lambda_f(n) = \\ = \begin{cases} \Delta_{2k,N}(m, n) - \frac{\Delta_{2k,N/p}(m, n)}{p-p^{-1}} & \text{if } \nu = 2 \text{ and } (N, mn) = 1, \\ \Delta_{2k,N}(m, n) - \frac{\Delta_{2k,N/p}(m, n)}{p} & \text{if } \nu \geq 3 \text{ and } (N, mn) = 1, \\ 0 & \text{if } (N, mn) = 1. \end{cases}$$

4. INTEGRALS INVOLVING THE BESSEL FUNCTIONS

Consider

$$(4.1) \quad I(z) = \int_0^\infty J_{2k-1}(y) k^+(y\sqrt{z}, 1/2 + it) dy,$$

where

$$(4.2) \quad k^+(y, v) = \frac{1}{2 \cos \pi v} (J_{2v-1}(y) - J_{1-2v}(y)).$$

Lemma 4.1. *([3], page 326) One has*

$$(4.3) \quad J_{2k-1}(y) = \frac{1}{4\pi i} \int_\Delta \frac{\Gamma(k - 1/2 + s/2)}{\Gamma(k + 1/2 - s/2)} \left(\frac{y}{2}\right)^{-s} ds$$

for some $1 - 2k < \Delta < 0$.

Lemma 4.2. *(equation 10.9.8 of [12]) For $|\Re v| < 1$ and $x > 0$*

$$(4.4) \quad J_v(x) = \frac{2}{\pi} \int_0^\infty \sin\left(x \cosh y - \frac{v\pi}{2}\right) \cosh(vy) dy.$$

Corollary 4.3. *For $a > 0$ and $v = it$ with $t \in \mathbf{R}$*

$$(4.5) \quad k^+(a, 1/2 + it) = \frac{2}{\pi} \int_0^\infty \cos(a \cosh z) \cos(2tz) dz = \\ = \frac{2}{\pi} \int_1^\infty \frac{\cos(au)}{\sqrt{u^2 - 1}} \cos\left(2t \log(u + \sqrt{u^2 - 1})\right) du.$$

Theorem 4.4. *For $z < 1$ the following decomposition takes place*

$$(4.6) \quad I(z) = \frac{2}{\pi} (Q_0(z) + (-1)^k Q_1(z)),$$

$$(4.7) \quad Q_0(z) = \int_1^{1/\sqrt{z}} \frac{\cos(2t \log(y + \sqrt{y^2 - 1}))}{\sqrt{1 - zy^2} \sqrt{y^2 - 1}} \cos[(2k - 1) \arcsin(y\sqrt{z})] dy,$$

$$(4.8) \quad Q_1(z) = \int_{1/\sqrt{z}}^{\infty} \frac{\cos(2t \log(y + \sqrt{y^2 - 1}))}{\sqrt{y^2 z - 1} \sqrt{y^2 - 1}} \frac{dy}{(y\sqrt{z} + \sqrt{y^2 z - 1})^{2k-1}}.$$

Corollary 4.5. *For $z < 1$ one has*

$$(4.9) \quad I(z) \ll 1 + |\log z| + \frac{1}{(2k - 1)\sqrt{1 - z}}.$$

Proof. Estimating (4.7) we obtain

$$Q_0(z) \ll \int_1^{1/\sqrt{z}} \frac{dy}{\sqrt{1 - zy^2} \sqrt{y^2 - 1}} = \frac{1}{\sqrt{z}} \int_1^{1/\sqrt{z}} \frac{dy}{\sqrt{1/z - y^2} \sqrt{y^2 - 1}}.$$

Applying equation 3.152(10) of [10] with $a = \frac{1}{\sqrt{z}}$, $b = 1$, $u = b$, $\lambda = \pi/2$

$$\frac{1}{\sqrt{z}} \int_1^{1/\sqrt{z}} \frac{dy}{\sqrt{1/z - y^2} \sqrt{y^2 - 1}} = F\left(\frac{\pi}{2} \sqrt{1 - z}\right),$$

where F is the elliptic integral defined by 8.111(2) of [10] or by 19.2.4 of [12]. Formulas 19.2.8 and 19.9.2 of [12] give

$$Q_0(z) \ll \log\left(\frac{4}{\sqrt{z}}\right) \left(1 + \frac{z}{4}\right) \ll \log \frac{4}{\sqrt{z}}.$$

Now consider (4.8)

$$Q_1(z) \ll \int_{1/\sqrt{z}}^{\infty} \frac{dy}{\sqrt{y^2 z - 1} \sqrt{y^2 - 1} (y\sqrt{z} + \sqrt{y^2 z - 1})^{2k-1}}.$$

Setting $r := y\sqrt{z}$ we have

$$\begin{aligned} Q_1(z) &\ll \frac{1}{\sqrt{z}} \int_1^{\infty} \frac{dr}{\sqrt{r^2 - 1} \sqrt{r^2/z - 1} (r + \sqrt{r^2 - 1})^{2k-1}} \ll \\ &\ll \frac{1}{\sqrt{1 - z}} \int_1^{\infty} \frac{dr}{\sqrt{r^2 - 1} (r + \sqrt{r^2 - 1})^{2k-1}} \ll \frac{1}{(2k - 1)\sqrt{1 - z}}. \end{aligned}$$

The last two estimates and theorem 4.4 yield the assertion. \square

4.1. Proof of theorem 4.4. We are going to evaluate the integral

$$(4.10) \quad I(k, t, 1/x) = \int_0^\infty J_{2k-1}(y) k^+ \left(\frac{y}{\sqrt{x}}, 1/2 + it \right) dy$$

for $x > 1$ using representation (4.5). However, in order to be able to change the order of integration in $I(k, t, 1/x)$, the uniform convergence of all integrals is required. To overcome this difficulty we make several preliminary steps. First, let us fix a large real number $Z > 0$ such that

$$(4.11) \quad k^+(a, 1/2 + it) = \frac{2}{\pi} \int_1^Z \frac{\cos(au)}{\sqrt{u^2 - 1}} \cos \left(2t \log(u + \sqrt{u^2 - 1}) \right) du + O \left(\frac{1 + |t|}{aZ} \right).$$

Second, we introduce an extra parameter $b \geq 0$ to avoid the discontinuity of the y -integral and consider

$$(4.12) \quad I_b(k, t, 1/x) = \int_0^\infty J_{2k-1}(y) k^+ \left(\frac{y}{\sqrt{x}}, 1/2 + it \right) \exp(-by) dy.$$

Note that

$$(4.13) \quad I(k, t, 1/x) = \lim_{b \rightarrow 0} I_b(k, t, 1/x).$$

Finally, for some $\epsilon > 0$ we split the integral over u into three parts

$$(4.14) \quad k^+(a, 1/2 + it) = \mathfrak{I}_1(a) + \mathfrak{I}_2(a) + \mathfrak{I}_3(a) + O \left(\frac{1 + |t|}{aZ} \right),$$

where

$$(4.15) \quad \mathfrak{I}_1(a) = \frac{2}{\pi} \int_1^{\sqrt{x}(1-\epsilon)} \frac{\cos(au)}{\sqrt{u^2 - 1}} \cos \left(2t \log(u + \sqrt{u^2 - 1}) \right) du,$$

$$(4.16) \quad \mathfrak{I}_2(a) = \frac{2}{\pi} \int_{\sqrt{x}(1-\epsilon)}^{\sqrt{x}(1+\epsilon)} \frac{\cos(au)}{\sqrt{u^2 - 1}} \cos \left(2t \log(u + \sqrt{u^2 - 1}) \right) du,$$

$$(4.17) \quad \mathfrak{I}_3(a) = \frac{2}{\pi} \int_{\sqrt{x}(1+\epsilon)}^Z \frac{\cos(au)}{\sqrt{u^2 - 1}} \cos \left(2t \log(u + \sqrt{u^2 - 1}) \right) du.$$

Next, we substitute equation (4.14) with $a := \frac{y}{\sqrt{x}}$ into $I_b(k, t, 1/x)$. The contribution of the error term in (4.14) into (4.12) is bounded by

$$\int_0^\infty |J_{2k-1}(y)| \frac{(1 + |t|)\sqrt{x}}{yZ} \exp(-by) dy \ll \frac{(1 + |t|)\sqrt{x}}{Z}.$$

Lemma 4.6. *The integral $\mathfrak{I}_1(y/\sqrt{x})$ contributes to $I(k, t, 1/x)$ as*

$$(4.18) \quad \frac{2}{\pi} Q_0(1/x) + O_k \left(\epsilon \sqrt{\frac{x}{x-1}} \right).$$

Proof. Consider

$$\begin{aligned} \mathfrak{L}_1 := & \frac{2}{\pi} \int_1^{\sqrt{x}(1-\epsilon)} \frac{\cos(2t \log(u + \sqrt{u^2 - 1}))}{\sqrt{u^2 - 1}} \times \\ & \times \left(\int_0^\infty \exp(-by) J_{2k-1}(y) \cos\left(\frac{u}{\sqrt{x}}y\right) dy \right) du. \end{aligned}$$

The inner integral

$$IY_1 := \int_0^\infty \exp(-by) J_{2k-1}(y) \cos\left(\frac{u}{\sqrt{x}}y\right) dy$$

can be evaluated using Mellin-Barnes representation (4.3). Changing the order of integration

$$\begin{aligned} IY_1 = & \frac{1}{4\pi i} \int_\Delta 2^s \frac{\Gamma(k - 1/2 + s/2)}{\Gamma(k + 1/2 - s/2)} \times \\ & \times \left(\int_0^\infty \exp(-by) \cos\left(\frac{u}{\sqrt{x}}y\right) y^{-s} dy \right) ds. \end{aligned}$$

For $\Re s < 1$ it follows that (see [9] pages 785 – 786)

$$\begin{aligned} \int_0^\infty \exp(-by) \cos\left(\frac{u}{\sqrt{x}}y\right) y^{-s} dy = \\ = \frac{\Gamma(1-s)}{(u^2/x + b^2)^{(1-s)/2}} \cos\left((1-s) \arctan \frac{u}{\sqrt{x}b}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} IY_1 = & \frac{1}{4\pi i} \int_\Delta \frac{\Gamma(k - 1/2 + s/2)}{\Gamma(k + 1/2 - s/2)} \Gamma(1-s) \times \\ & \times \cos\left((1-s) \arctan \frac{u}{\sqrt{x}b}\right) \frac{2^s}{(u^2/x + b^2)^{(1-s)/2}} ds. \end{aligned}$$

Since $b \rightarrow 0$ and $u/\sqrt{x} \leq 1 - \epsilon$, we have $u^2/x + b^2 \leq 1 - \epsilon/2$. Thus the expression under the integral is a rapidly decreasing function as $s \rightarrow \infty$. Moving the contour of integration to the right, we cross poles at $s_j = 1 + j$, $j = 0, 1, \dots$. Hence

$$IY_1 = \frac{1}{2} \sum_{j=0}^\infty \frac{\Gamma(k + j/2)}{\Gamma(k - j/2)} \frac{(-1)^j}{j!} \cos\left(j \arctan \frac{u}{\sqrt{x}b}\right) \frac{2^{1+j}}{(u^2/x + b^2)^{-j/2}}.$$

Now we can substitute IY_1 into \mathfrak{L}_1 and compute the limit as $b \rightarrow 0$. Uniform convergence on b follows by Weierstrass M-Test since $u^2/x +$

$b^2 \leq 1 - \epsilon/2$. Thus it is possible to interchange the order of summation and integration

$$\begin{aligned} \mathfrak{L}_1 &= \frac{2}{\pi} \int_1^{\sqrt{x}(1-\epsilon)} \frac{\cos(2t \log(u + \sqrt{u^2 - 1}))}{\sqrt{u^2 - 1}} \times \\ &\quad \times \sum_{j=0}^{\infty} \frac{\Gamma(k + j/2)}{\Gamma(k - j/2)} \frac{(-1)^j \cos(\pi j/2)}{j!} \frac{2^j du}{(u/\sqrt{x})^{-j}} = \\ &= \frac{2}{\pi} \int_1^{\sqrt{x}(1-\epsilon)} \frac{\cos(2t \log(u + \sqrt{u^2 - 1}))}{\sqrt{u^2 - 1}} \times \\ &\quad \times \sum_{m=0}^{k-1} \frac{\Gamma(k + m)}{\Gamma(k - m)} \frac{(-1)^m}{(2m)!} \left(\frac{2u}{\sqrt{x}}\right)^{2m} du. \end{aligned}$$

By formula 1.332(4) of [10]

$$\frac{\cos((2k-1) \arcsin c)}{\sqrt{1-c^2}} = (-1)^k \sum_{n=1}^k \frac{\Gamma(2k-n)}{\Gamma(n)(2k-2n)!} (-1)^n c^{2k-2n}.$$

Therefore,

$$\mathfrak{L}_1 = \frac{2}{\pi} Q_0(1/x) - E_1,$$

where $Q_0(x)$ is defined by (4.7) and

$$\begin{aligned} E_1 &:= \frac{2}{\pi} \int_{\sqrt{x}(1-\epsilon)}^{\sqrt{x}} \frac{\cos(2t \log(u + \sqrt{u^2 - 1}))}{\sqrt{u^2 - 1}} \times \\ &\quad \times \sum_{m=0}^{k-1} \frac{\Gamma(k + m)}{\Gamma(k - m)} \frac{(-1)^m}{(2m)!} \left(\frac{2u}{\sqrt{x}}\right)^{2m} du. \end{aligned}$$

The error term can be estimated as follows

$$E_1 \ll_k \frac{\epsilon \sqrt{x}}{\sqrt{x-1}}.$$

□

Lemma 4.7. *The integral $\mathfrak{I}_2(y/\sqrt{x})$ contributes to $I(k, t, 1/x)$ as*

$$(4.19) \quad O\left(\epsilon^{1/3} \sqrt{\frac{x}{x-1}} + \epsilon \frac{x}{x-1} \left(t + \sqrt{\frac{x}{x-1}}\right)\right).$$

Proof. Integrating by parts we have

$$\mathfrak{I}_2(a) = \mathfrak{I}_{2,1}(a) - \mathfrak{I}_{2,2}(a),$$

where

$$\mathfrak{I}_{2,1}(a) := \frac{\sin au \cos(2t \log(u + \sqrt{u^2 - 1}))}{a \sqrt{u^2 - 1}} \Big|_{\sqrt{x}(1-\epsilon)}^{\sqrt{x}(1+\epsilon)},$$

$$\mathfrak{I}_{2,2}(a) := \int_{\sqrt{x}(1-\epsilon)}^{\sqrt{x}(1+\epsilon)} \frac{\sin au}{a} \left(\frac{\cos(2t \log(u + \sqrt{u^2 - 1}))}{\sqrt{u^2 - 1}} \right)' du.$$

Using

$$\left(\frac{\cos(2t \log(u + \sqrt{u^2 - 1}))}{\sqrt{u^2 - 1}} \right)' \ll \frac{t}{u^2 - 1} + \frac{u}{(u^2 - 1)^{3/2}},$$

we obtain

$$\mathfrak{I}_{2,2}(a) \ll \frac{\epsilon \sqrt{x}}{a} \left(\frac{t}{x - 1} + \frac{\sqrt{x}}{(x - 1)^{3/2}} \right).$$

On the one hand, it is possible to estimate the non-integral term trivially

$$\mathfrak{I}_{2,1}(a) \ll \frac{1}{a \sqrt{x - 1}}.$$

On the other hand, we can apply

$$\begin{aligned} \left(\frac{\sin au \cos(2t \log(u + \sqrt{u^2 - 1}))}{a \sqrt{u^2 - 1}} \right)' &\ll \\ &\ll \frac{1}{(u^2 - 1)^{1/2}} + \frac{t}{a(u^2 - 1)} + \frac{u}{a(u^2 - 1)^{3/2}}. \end{aligned}$$

Then the mean value theorem gives

$$\mathfrak{I}_{2,1}(a) \ll \epsilon \sqrt{x} \left(\frac{1}{\sqrt{x - 1}} + \frac{t}{a(x - 1)} + \frac{\sqrt{x}}{a(x - 1)^{3/2}} \right).$$

Combining all the results

$$\mathfrak{I}_2(a) \ll \frac{\epsilon \sqrt{x}}{\sqrt{x - 1}} \min \left(1, \frac{1}{a \epsilon \sqrt{x}} \right) + \frac{\epsilon \sqrt{x}}{a(x - 1)} \left(t + \sqrt{\frac{x}{x - 1}} \right).$$

Estimating

$$\frac{\epsilon \sqrt{x}}{\sqrt{x - 1}} \min \left(1, \frac{1}{a \epsilon \sqrt{x}} \right) \ll \frac{(\epsilon \sqrt{x})^{1/3}}{a^{2/3} \sqrt{x - 1}}$$

and setting $a := y/\sqrt{x}$ we have

$$\mathfrak{I}_2(y/\sqrt{x}) \ll \frac{\epsilon^{1/3} \sqrt{x}}{y^{2/3} \sqrt{x - 1}} + \frac{\epsilon}{y} \frac{x}{x - 1} \left(t + \sqrt{\frac{x}{x - 1}} \right).$$

Therefore, $\mathfrak{I}_2(y/\sqrt{x})$ contributes to $I(k, t, 1/x)$ as

$$O\left(\epsilon^{1/3}\sqrt{\frac{x}{x-1}} + \epsilon\frac{x}{x-1}\left(t + \sqrt{\frac{x}{x-1}}\right)\right).$$

□

Lemma 4.8. *The integral $\mathfrak{I}_3(y/\sqrt{x})$ contributes to $I(k, t, 1/x)$ as*

$$(4.20) \quad \frac{2}{\pi}(-1)^k Q_1(1/x) + O\left(\epsilon^{1/2}\sqrt{\frac{x}{x-1}} + \frac{x^k}{Z^{2k}}\right).$$

Proof. We need to evaluate the following expression as $b \rightarrow 0$

$$\begin{aligned} \mathfrak{L}_2 &:= \frac{2}{\pi} \int_{\sqrt{x}(1+\epsilon)}^Z \frac{\cos(2t \log(u + \sqrt{u^2 - 1}))}{\sqrt{u^2 - 1}} \times \\ &\quad \times \left(\int_0^\infty \exp(-by) J_{2k-1}(y) \cos\left(\frac{u}{\sqrt{x}}y\right) dy \right) du. \end{aligned}$$

Calculating the y -integral in the same way as in lemma 4.6 we obtain

$$\begin{aligned} IY_2 &= \frac{1}{4\pi i} \int_{\Delta} \frac{\Gamma(k - 1/2 + s/2)}{\Gamma(k + 1/2 - s/2)} \Gamma(1 - s) \times \\ &\quad \times \cos\left((1 - s) \arctan \frac{u}{\sqrt{xb}}\right) \frac{2^s}{(u^2/x + b^2)^{(1-s)/2}} ds. \end{aligned}$$

Note that now $u^2/x + b^2 > 1$ and, therefore, we move the contour of integration to the left crossing poles at $s = 1 - 2k - 2j$, $j = 0, 1, \dots$. This yields

$$\begin{aligned} IY_2 &= \sum_{j=0}^{\infty} \frac{\Gamma(2k + 2j)}{\Gamma(2k + j)} \frac{(-1)^j}{j!} \times \\ &\quad \times \cos\left((2k + 2j) \arctan \frac{u}{\sqrt{xb}}\right) \frac{2^{1-2k-2j}}{(u^2/x + b^2)^{k+j}}. \end{aligned}$$

Substituting this into \mathfrak{L}_2 and letting $b \rightarrow 0$ gives

$$\begin{aligned} \mathfrak{L}_2 &:= \frac{2}{\pi} \int_{\sqrt{x}(1+\epsilon)}^Z \frac{\cos(2t \log(u + \sqrt{u^2 - 1}))}{\sqrt{u^2 - 1}} \times \\ &\quad \times \sum_{j=0}^{\infty} \frac{\Gamma(2k + 2j)}{\Gamma(2k + j)} \frac{(-1)^j}{j!} \frac{2^{1-2k-2j}}{(u/\sqrt{x})^{2k+2j}} du. \end{aligned}$$

Recall that

$$\Gamma(2k + 2j) = \frac{1}{\sqrt{\pi}} 2^{2k+2j-1} \Gamma(k + j) \Gamma(k + j + 1/2)$$

and

$$\begin{aligned} \frac{1}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{\Gamma(k+j)\Gamma(k+j+1/2)}{\Gamma(j+2k)j!} \frac{(-1)^k}{c^{2j+2k}} &= \frac{(-1)^k}{c^{2k}\sqrt{\pi}} \frac{\Gamma(k)\Gamma(k+1/2)}{\Gamma(2k)} \times \\ &\times {}_2F_1(k, k+1/2, 2k; 1/c^2) = \frac{(-1)^k}{\sqrt{c^2-1}(c+\sqrt{c^2-1})^{2k-1}}. \end{aligned}$$

The last equality follows by formula 2.8(6) of [2]. This yields

$$\begin{aligned} \mathfrak{L}_2 &= \frac{2}{\pi} \int_{\sqrt{x}(1+\epsilon)}^Z \frac{\cos(2t \log(u + \sqrt{u^2-1}))}{\sqrt{u^2-1}} \times \\ &\times \frac{(-1)^k du}{\sqrt{u^2/x-1} (u/\sqrt{x} + \sqrt{u^2/x-1})^{2k-1}}. \end{aligned}$$

Finally we split the integral into three parts

$$\int_{\sqrt{x}(1+\epsilon)}^Z = \int_{\sqrt{x}(1+\epsilon)}^{\infty} - \int_{\sqrt{x}}^{\sqrt{x}(1+\epsilon)} - \int_Z^{\infty}.$$

The first part contributes as $\frac{2}{\pi}(-1)^k Q_1(1/x)$ and the last two as

$$O\left(\epsilon^{1/2} \sqrt{\frac{x}{x-1}} + \frac{x^k}{Z^{2k}}\right).$$

This completes the proof. \square

Letting $Z \rightarrow \infty$, $\epsilon \rightarrow 0$ in lemmas 4.6, 4.7 and 4.8, we obtain the statement of theorem 4.4.

5. HYPERGEOMETRIC FUNCTIONS

Let $\Re\lambda > 1 + \Re u \geq 1$, $\Re v = 0$, $\Im v = t$ with $t \in \mathbf{R}$ and $T := 3 + |t|$. The main object of this section is the function defined via two integrals

$$(5.1) \quad H_{\lambda}(u, v, z) = \begin{cases} 1/(2 \cos \pi(\lambda - u)) z^{1/2-\lambda} I_1(1/z) & z > 0 \\ 1/2(-z)^{1/2-\lambda} I_2(-1/z) & z < 0 \end{cases},$$

where

$$\begin{aligned} I_1(z) &= \frac{1}{2\pi i} \int_{\Re s = \Delta} \Gamma(u, v, \lambda; s) \sin \pi \left(u + \frac{s}{2}\right) z^{s/2} ds, \\ I_2(z) &= \frac{1}{2\pi i} \int_{\Re s = \Delta} \Gamma(u, v, \lambda; s) z^{s/2} ds \end{aligned}$$

with $1 - 2\Re\lambda < \Delta < -1 - 2\Re u$.

Lemma 5.1. For $z \leq 1$, $z \neq 0$, $\Re u \geq 0$

$$(5.2) \quad H_\lambda(u, v; z) = \frac{\Gamma(\lambda - u + v)\Gamma(\lambda - u - v)}{\Gamma(2\lambda)} {}_2F_1(\lambda - u + v, \lambda - u - v, 2\lambda; z).$$

Proof. This formula follows by moving the contour of integration to the left and computing the residues. \square

Lemma 5.2. ([8]) For $z > 0$, $k \geq 1$

$$(5.3) \quad \cosh(\pi t) H_k(0, v, -z) \ll \frac{T^{2k-1}}{2^{2k}\sqrt{k}} \left[1 + O\left(z^{1/2} \frac{k+T}{\sqrt{k}}\right) \right].$$

Lemma 5.3. ([7], page 3) For $z > 0$, $k \geq 1$

$$(5.4) \quad \cosh(\pi t) H_k(0, v, -z) \ll z^{-k+1/2}.$$

Lemma 5.4. For $z > 0$, $k \geq 1$

$$(5.5) \quad \cosh(\pi t) H_k(0, v, -z) \ll \frac{z^{-k}}{\sqrt{T}} \left[1 + \log(zk) + O\left(z^{-1/2} \frac{k+T}{\sqrt{T}}\right) \right].$$

Proof. The Mellin-Barnes representation(formula 15.6.6 of [12]) gives

$$(5.6) \quad H_k(0, v, -z) = \frac{1}{2\pi i} \int_{\Re s=c} \frac{\Gamma(k+it+s)\Gamma(k-it+s)\Gamma(-s)}{\Gamma(2k+s)} x^s ds$$

for $-k < c < 0$. Moving the contour of integration to $-k - 1/2$ we have

$$\begin{aligned} H_k(0, v, -z) &= z^{-k+it} \Gamma(2it) \frac{\Gamma(k-it)}{\Gamma(k+it)} + \\ &\quad + z^{-k-it} \Gamma(-2it) \frac{\Gamma(k+it)}{\Gamma(k-it)} + O(z^{-k-1/2} J), \end{aligned}$$

where

$$J = \int_{-\infty}^{+\infty} \left| \frac{\Gamma(-1/2 + i(r+t))\Gamma(-1/2 + i(r-t))\Gamma(k+1/2 - ir)}{\Gamma(k-1/2 + ir)} \right| dr.$$

The last integral can be estimated using the standard identities for the Gamma function

$$J \ll \int_{-\infty}^{+\infty} \frac{\sqrt{k^2 + r^2} (\cosh 2\pi r + \cosh 2\pi t)^{-1/2}}{\sqrt{1/4 + (r-t)^2} \sqrt{1/4 + (r+t)^2}} dr \ll \frac{k+t}{T} \frac{\log T}{\exp(\pi t)}.$$

Therefore, for $t \neq 0$

$$\cosh \pi t H_k(0, v, -z) \ll \frac{z^{-k}}{\sqrt{T}} \left(1 + O \left(\frac{z+t}{\sqrt{zT}} \right) \right).$$

Computing the limit as $t \rightarrow 0$

$$H_k(0, 0, -z) = z^{-k} (\log z - 2\gamma - 2\psi(k)) + O(z^{-k-1/2}k).$$

The result follows from the last two estimates. \square

Lemma 5.5. *For $0 < z < 1$, $k \geq 1$*

$$(5.7) \quad z^k H_k(0, v, z) \ll \sqrt{\frac{z}{1-z}} \frac{1}{k+t}.$$

Proof. Using the relation

$${}_1F_2(a, b, c; z) = (1-z)^{-a} {}_1F_2(a, c-b, c; z/(z-1))$$

we have

$$\begin{aligned} z^k H_k(0, v, z) &= \frac{\Gamma(k-v)\Gamma(k+v)}{\Gamma(2k)} \frac{z^k}{(1-z)^{k-v}} \times \\ &\times {}_1F_2 \left(k-v, k-v, 2k; -\frac{z}{1-z} \right) = \frac{\Gamma(k+v)}{\Gamma(k-v)} \frac{z^k}{(1-z)^{k-v}} \times \\ &\times \frac{1}{2\pi i} \int_{\Re s=c} \frac{\Gamma^2(k-v+s)\Gamma(-s)}{\Gamma(2k+s)} \left(\frac{z}{1-z} \right)^s ds, \end{aligned}$$

where $-k < c < 0$. Moving the contour of integration to $c = -k + 1/2$ we obtain

$$z^k H_k(0, v, z) \ll \sqrt{\frac{z}{1-z}} \int_{-\infty}^{+\infty} \frac{dr}{\cosh \pi(r-t) \sqrt{r^2 + (k-1/2)^2}}.$$

Next, we estimate the last integral by splitting it into two parts

$$\int_{-\infty}^{+\infty} = \int_{-\infty}^0 + \int_0^{+\infty}.$$

Let $t \geq 0$ then

$$\int_{-\infty}^0 \frac{dr}{\cosh \pi(r-t) \sqrt{r^2 + (k-1/2)^2}} \ll \frac{\exp(-\pi t)}{k}.$$

For $t < k - 1/2$

$$\int_0^{+\infty} \frac{dr}{\cosh \pi(r-t) \sqrt{r^2 + (k-1/2)^2}} \ll \frac{1}{k}$$

and for $t > k - 1/2$

$$\int_0^{+\infty} \frac{dr}{\cosh \pi(r-t) \sqrt{r^2 + (k-1/2)^2}} \ll \frac{1}{t}.$$

This yields the result. \square

Lemma 5.6. ([7], page 2) For $0 < z \leq 1/2$, $k \geq 1$

$$(5.8) \quad z^k H_k(0, v, z) \ll \frac{z^k}{2^k}.$$

Lemma 5.7. Let $\Re v = 0$ and $\Re \lambda > \Re u > 0$. For $0 < z < 1$

$$(5.9) \quad \begin{aligned} 2 \cos \pi(\lambda - u) H_\lambda(u, v, 1/z) &= \\ &= 2\pi 2^{2u} z^{\lambda-u} \int_0^\infty J_{2\lambda-1}(x) k^+(x\sqrt{z}, 1/2 + v) x^{-2u} dx. \end{aligned}$$

Proof. By definition

$$2 \cos \pi(\lambda - u) H_\lambda(u, v, 1/z) = z^{\lambda-1/2} I_1(z).$$

Representation (4.3) yields that the Mellin transform of J -Bessel function equals

$$\widehat{J}_{2\lambda-1}(s) = 2^{s-1} \frac{\Gamma(\lambda - 1/2 + s/2)}{\Gamma(\lambda + 1/2 - s/2)}, \quad 1 - 2\lambda < \Re s < 3/2.$$

Let

$$\gamma(1/2 - u - s/2, 1/2 + v) := \frac{2^{-2u-s}}{\pi} \Gamma(1/2 - u - s/2 + v) \Gamma(1/2 - u - s/2 - v),$$

$$g_1(x) =: x^{-2u} k^+(x\sqrt{z}, 1/2 + v),$$

where k^+ is defined by (4.2). Then

$$z^{s/2} \gamma(1/2 - u - s/2, 1/2 + v) \sin \pi(u + s/2) = z^{1/2-u} \widehat{g}_1(1-s).$$

Therefore,

$$I_1(z) = z^{1/2-u} \frac{2\pi 2^{2u}}{2\pi i} \int_{\Re s = \Delta} \widehat{J}_{2\lambda-1}(s) \widehat{g}_1(1-s) ds.$$

Note that for $1/2 - 2\Re u < \Delta < 1 - 2u$

$$\int_0^\infty |g_1(x) x^{-\Delta}| dx < \infty$$

and for $\Delta < 0$ we have

$$\int_{\Re s = \Delta} \left| \widehat{J}_{2\lambda-1}(s) \right| ds < \infty.$$

Therefore, for $\Re u > 1/4$ and $\Re v = 0$

$$I_1(z) = 2\pi 2^{2u} z^{1/2-u} \int_0^\infty g_1(x) J_{2\lambda-1}(x) dx.$$

Since both parts of equation (5.9) are analytic functions in the larger region $\Re \lambda > \Re u > 0$ the result extends to this region. \square

6. THE SECOND MOMENT: $\nu \geq 3$

We prove theorem 1.1 for $k \geq 2$. The most delicate and technical case $k = 1$ follows by combining our results with the methods developed in [6]. To keep the length of the paper reasonable, we omit the details and note that the case $k = 1$ requires just minor modifications comparing to [6].

Lemma 6.1. *Suppose that $\Re u > 1/2$, $\Re v = 0$, $k \geq 2$, $N = p^\nu$ and $\nu \geq 3$. Then*

$$(6.1) \quad M_2(l, u, v) = S(l, u, v; N) - \frac{1}{p} S(l, u, v; N/p),$$

where

$$(6.2) \quad S(l, u, v; N) = \mathbf{1}_{(l,p)=1} \frac{1}{l^{1/2+u-v}} \sum_{d|l} d^{1/2+u-v} \sum_{\substack{m,n=1 \\ (mn,p)=1}}^{\infty} \frac{(m/n)^v}{(mn)^{1/2+u}} \Delta_{2k,N}(md, n).$$

Proof. It follows from definition that

$$M_2(l, u, v) = \sum_{m,n=1}^{\infty} \frac{1}{(mn)^{1/2+u}} \left(\frac{m}{n}\right)^v \sum_{f \in H_{2k}^*(N)}^h \lambda_f(l) \lambda_f(m) \lambda_f(n).$$

Property of multiplicity (3.6) yields

$$\begin{aligned} M_2(l, u, v) &= \sum_{\substack{d|l \\ (d,p)=1}} \sum_{m \equiv 0 \pmod{d}} \sum_{n=1}^{\infty} \frac{1}{(mn)^{1/2+u}} \left(\frac{m}{n}\right)^v \Delta_{2k,N}^* \left(\frac{ml}{d^2}, n\right) = \\ &= \sum_{\substack{d|l \\ (d,p)=1}} \frac{1}{d^{1/2+u-v}} \sum_{m,n=1}^{\infty} \frac{1}{(mn)^{1/2+u}} \left(\frac{m}{n}\right)^v \Delta_{2k,N}^* \left(\frac{ml}{d}, n\right). \end{aligned}$$

According to theorem 3.2

$$\Delta_{2k,N}^* \left(\frac{ml}{d}, n\right) \neq 0 \text{ only if } \left(\frac{ml}{d} n, p\right) = 1.$$

Hence $\left(\frac{l}{d}, p\right) = 1$. Since $(d, p) = 1$, we have $(l, p) = 1$. Note that the condition $(d, p) = 1$ in summation over d is satisfied if $(l, p) = 1$ because $d|l$. Therefore,

$$\begin{aligned} M_2(l, u, v) &= \sum_{d|l} \frac{\mathbf{1}_{(l,p)=1}}{d^{1/2+u-v}} \sum_{\substack{m,n=1 \\ (mn,p)=1}}^{\infty} \frac{1}{(mn)^{1/2+u}} \left(\frac{m}{n}\right)^v \Delta_{2k,N}^* \left(\frac{ml}{d}, n\right) = \\ &= \frac{\mathbf{1}_{(l,p)=1}}{l^{1/2+u-v}} \sum_{d|l} d^{1/2+u-v} \sum_{\substack{m,n=1 \\ (mn,p)=1}}^{\infty} \frac{1}{(mn)^{1/2+u}} \left(\frac{m}{n}\right)^v \Delta_{2k,N}^* (md, n). \end{aligned}$$

Applying theorem 3.2 we obtain the assertion. \square

Define

$$(6.3) \quad T_1(l, u, v; N) = \sum_{m,n=1}^{\infty} \frac{1}{(mn)^{1/2+u}} \left(\frac{m}{n}\right)^v \Delta_{2k,N} (md, n),$$

$$(6.4) \quad T_2(l, u, v; N) = \sum_{m,n=1}^{\infty} \frac{1}{(pmn)^{1/2+u}} \left(\frac{pm}{n}\right)^v \Delta_{2k,N} (pmd, n),$$

$$(6.5) \quad T_3(l, u, v; N) = \sum_{\substack{m,n=1 \\ (m,p)=1}}^{\infty} \frac{1}{(pmn)^{1/2+u}} \left(\frac{m}{pn}\right)^v \Delta_{2k,N} (md, pn)$$

and let for $i = 1, 2, 3$

$$(6.6) \quad S_i(l, u, v; N) = \frac{\mathbf{1}_{(l,p)=1}}{l^{1/2+u-v}} \sum_{d|l} d^{1/2+u-v} T_i(l, u, v; N).$$

Lemma 6.2. *The following decomposition takes place*

$$(6.7) \quad S(l, u, v; N) = S_1(l, u, v; N) - S_2(l, u, v; N) - S_3(l, u, v; N).$$

Proof. Note that

$$\sum_{\substack{m,n=1 \\ (mn,p)=1}}^{\infty} f(m, n) = \sum_{m,n=1}^{\infty} f(m, n) - \sum_{m,n=1}^{\infty} f(pm, n) - \sum_{\substack{m,n=1 \\ (m,p)=1}}^{\infty} f(m, pn).$$

Applying this equality to $S(l, u, v; N)$ yields the result. \square

Lemma 6.3. *The sums $S_3(l, u, v; N)$ and $S_3(l, u, v; N/p)$ vanish.*

Proof. Consider

$$\begin{aligned} \Delta_{2k,N}(dm, pn) &= \delta(dm, pn) + \\ &+ 2\pi i^{-2k} \sum_{q \equiv (\text{mod } N)} \frac{Kl(dm, pn; q)}{q} J_{2k-1} \left(4\pi \frac{\sqrt{dmpn}}{q} \right). \end{aligned}$$

Conditions $(d, p) = 1$ and $(m, p) = 1$ imply that $\delta(dm, pn) = 0$. Since $q \equiv 0 \pmod{p^2}$, lemma 2.1 yields that $Kl(dm, pn; q) = 0$ and, therefore, $T_3(l, u, v; N) = 0$. \square

We introduce

$$\begin{aligned} (6.8) \quad D_N^*(u, v; \lambda) &= \sum_{m,n=1}^{\infty} \frac{1}{(mn)^{1/2+u}} \left(\frac{m}{n} \right)^v \times \\ &\times \sum_{q \equiv 0 \pmod{N}} \frac{Kl(dm, n; q)}{q} J_{2\lambda-1} \left(4\pi \frac{\sqrt{dmn}}{q} \right). \end{aligned}$$

Petersson's trace formula 3.1 gives the following representation for (6.3).

Lemma 6.4. *Let $\Re u > 3/4$ and $\Re v = 0$. Then*

$$(6.9) \quad T_1(l, u, v; N) = \frac{\zeta(1+2u)}{d^{1/2+u+v}} + 2\pi i^{2k} D_N^*(u, v; \lambda).$$

For $\Re s > 1 + |\Re v|$ let

$$(6.10) \quad G^*(s, v; d, q) = \sum_{m,n=1}^{\infty} \frac{Kl(md, n; q)}{(mn)^s} \left(\frac{m}{n} \right)^v.$$

This is a generalization of function $G(s, v; q)$ given on page 5 of [6], namely $G^*(s, v; 1, q) = G(s, v; q)$.

Lemma 6.5. *For $q \in \mathbf{N}$ and $v \in \mathbf{C}$ the function $G^*(s, v; d, q)$ can be analytically continued on the whole complex plane as a function of*

complex variable s . Furthermore, for $\Re s < -|\Re v|$ one has

$$(6.11) \quad G^*(s, v; d, q) = 2\Gamma(1-s+v)\Gamma(1-s-v) \left(\frac{2\pi}{q}\right)^{2s-2} \times \\ \times \left(-\cos \pi s \sum_{\substack{m,n=1 \\ (n,q)=1}}^{\infty} \frac{\delta_q(mn-d)}{(mn)^{1-s}} \left(\frac{m}{n}\right)^{-v} + \right. \\ \left. + \cos \pi v \sum_{\substack{m,n=1 \\ (n,q)=1}}^{\infty} \frac{\delta_q(mn+d)}{(mn)^{1-s}} \left(\frac{m}{n}\right)^{-v} \right).$$

Proof. This can be proved analogously to lemma 2.2 of [6] using the identity below

$$\sum_{\substack{a,b=0 \\ ad \equiv m \pmod{q} \\ b \equiv n \pmod{q}}}^{q-1} \delta_q(ab-1) = \sum_{\substack{a=0 \\ ad \equiv m \pmod{q}}}^{q-1} \delta_q(an-1) = \delta_q(mn-d) \mathbf{1}_{(n,q)=1}.$$

□

Applying lemma 6.5 and following the arguments of lemmas 3.1-3.4 of [6] we obtain

Lemma 6.6. *Let $\Re \lambda - 1 > \Re u > 3/4$ and $\Re v = 0$. Then*

$$(6.12) \quad D_N^*(u, v; \lambda) = \frac{(2\pi)^{2u-1}}{2\pi i} \int_{\Re s = \Delta} \Gamma(u, v, \lambda; s) \times \\ \times \left(\sin \pi \left(u + \frac{s}{2}\right) \sum_{q \equiv 0 \pmod{N}} \sum_{\substack{m,n=1 \\ (n,q)=1}}^{\infty} \frac{\delta_q(mn-d)}{q^{2u}(mn)^{1/2-u-s/2}} \left(\frac{m}{n}\right)^{-v} + \right. \\ \left. + \cos \pi v \sum_{q \equiv 0 \pmod{N}} \sum_{\substack{m,n=1 \\ (n,q)=1}}^{\infty} \frac{\delta_q(mn+d)}{q^{2u}(mn)^{1/2-u-s/2}} \left(\frac{m}{n}\right)^{-v} \right) \frac{ds}{d^{s/2}},$$

where $1 - 2\Re \lambda < \Delta < -1 - 2\Re u$.

Lemma 6.7. *For $\Re \lambda - 1 > \Re u > 3/4$ and $\Re v = 0$ one has*

$$\begin{aligned}
 (6.13) \quad D_N^*(u, v; \lambda) &= \frac{(2\pi)^{2u-1}}{2\pi i N^{2u}} \int_{\Re s = \Delta} \Gamma(u, v, \lambda; s) \times \\
 &\quad \times \left(\zeta(2u) \sin \pi \left(u + \frac{s}{2} \right) \sum_{r|d} \frac{\mu(d/r)}{d^{1/2+u-v-s/2}} \frac{\tau_v(r)}{r^{v-2u}} + \right. \\
 &\quad + \sin \pi \left(u + \frac{s}{2} \right) \sum_{r|d} \frac{\mu(d/r)}{(d/r)^{1/2+u-v-s/2}} \sum_{\substack{n \geq (1-r)/N \\ n \neq 0}} \frac{\tau_u(|n|) \tau_v(nN+r)}{|n|^u (nN+r)^{1/2-u-s/2}} + \\
 &\quad \left. + \cos \pi v \sum_{r|d} \frac{\mu(d/r)}{(d/r)^{1/2+u-v-s/2}} \sum_{n \geq (1+r)/N} \frac{\tau_u(n) \tau_v(nN-r)}{n^u (nN-r)^{1/2-u-s/2}} \right) \frac{ds}{d^{s/2}},
 \end{aligned}$$

where $1 - 2\Re \lambda < \Delta < -1 - 2\Re u$.

Proof. Consider

$$\begin{aligned}
 P(z) &:= \sum_{q \equiv 0 \pmod{N}} \sum_{\substack{m, n=1 \\ (n, q)=1}}^{\infty} \frac{\delta_q(mn-d)}{q^{2u}(mn)^z} \left(\frac{m}{n} \right)^{-v} = \\
 &= \frac{1}{N^{2u}} \sum_{q=1}^{\infty} \sum_{\substack{m, n=1 \\ (n, qN)=1}}^{\infty} \frac{\delta_{qN}(mn-d)}{q^{2u}(mn)^z} \left(\frac{m}{n} \right)^{-v}.
 \end{aligned}$$

Recall that $(d, N) = 1$. Hence the condition $mn \equiv d \pmod{qN}$ implies that $(n, N) = 1$. Therefore, $(n, qN) = 1$ can be replaced by $(n, q) = 1$ in the sum over n . We remove the last coprimality condition by Möbius inversion

$$\begin{aligned}
 P(z) &= \frac{1}{N^{2u}} \sum_{m, n, q=1}^{\infty} \frac{\delta_{qN}(mn-d)}{q^{2u}(mn)^z} \left(\frac{m}{n} \right)^{-v} \sum_{k|(n, q)} \mu(k) = \\
 &= \frac{1}{N^{2u}} \sum_{k=1}^{\infty} \mu(k) \sum_{q \equiv 0 \pmod{k}} \frac{1}{q^{2u}} \sum_{\substack{m, n=1 \\ n \equiv 0 \pmod{k}}}^{\infty} \frac{\delta_{qN}(mn-d)}{(mn)^z} \left(\frac{m}{n} \right)^{-v} = \\
 &= \frac{1}{N^{2u}} \sum_{k=1}^{\infty} \frac{\mu(k)}{k^{2u+z-v}} \sum_{q=1}^{\infty} \frac{1}{q^{2u}} \sum_{m, n=1}^{\infty} \frac{\delta_{qkN}(mnk-d)}{(mn)^z} \left(\frac{m}{n} \right)^{-v}.
 \end{aligned}$$

It follows from $mnk \equiv d \pmod{qNk}$ that $k|d$. The change of variables $r := \frac{d}{k}$ gives

$$P(z) = \frac{1}{N^{2u}} \sum_{r|d} \frac{\mu(d/r)}{(d/r)^{2u+z-v}} \sum_{q=1}^{\infty} \frac{1}{q^{2u}} \sum_{m,n=1}^{\infty} \frac{\delta_{qN}(mn-r)}{(mn)^z} \left(\frac{m}{n}\right)^{-v}.$$

Let $mn := r + aN$ with $a \in \mathbf{Z}$, then

$$\begin{aligned} \sum_{q=1}^{\infty} \frac{1}{q^{2u}} \sum_{m,n=1}^{\infty} \frac{\delta_{qN}(mn-r)}{(mn)^z} \left(\frac{m}{n}\right)^{-v} &= \\ &= \zeta(2u) \frac{\tau_v(r)}{r^z} + \sum_{q=1}^{\infty} \sum_{\substack{a \geq (1-r)/N \\ a \neq 0}} \frac{\delta_{qN}(aN)}{q^{2u}(r+aN)^z} \sum_{mn=r+aN} \left(\frac{m}{n}\right)^{-v} = \\ &= \zeta(2u) \frac{\tau_v(r)}{r^z} + \sum_{\substack{a \geq (1-r)/N \\ a \neq 0}} \frac{\tau_v(r+aN)\tau_u(|a|)}{(r+aN)^z |a|^u}. \end{aligned}$$

Analogously,

$$\sum_{q=1}^{\infty} \frac{1}{q^{2u}} \sum_{m,n=1}^{\infty} \frac{\delta_{qN}(mn+r)}{(mn)^z} \left(\frac{m}{n}\right)^{-v} = \sum_{a \geq (1+r)/N} \frac{\tau_v(aN-r)\tau_u(a)}{(aN-r)^z a^u}.$$

Applying the last three formulas to (6.12), we obtain the assertion. \square

Lemma 6.8. *For $\Re \lambda > 1$, $\Re u = 0$, $\Re v = 0$ and $u \neq 0$*

$$\begin{aligned} (6.14) \quad D_N^*(u, v; \lambda) &= \frac{2(2\pi)^{2u-1}}{N^{2u}} \zeta(2u) \Gamma(2u) \cos \pi(\lambda - u) \times \\ &\times \frac{\Gamma(\lambda - u + v) \Gamma(\lambda - u - v)}{\Gamma(\lambda + u + v) \Gamma(\lambda + u - v)} \sum_{r|d} \frac{\mu(d/r)}{d^{1/2+u-v}} \frac{\tau_v(r)}{r^{v-2u}} + \\ &+ \frac{2(2\pi)^{2u-1}}{N^{2u}} \sum_{r|d} \frac{\mu(d/r)}{d^{1/2+u-v} r^{-\lambda-u+v}} \times \\ &\times \left(\cos \pi(\lambda - u) \sum_{\substack{n \geq (1-r)/N \\ n \neq 0}} \frac{\tau_u(|n|)\tau_v(nN+r)}{|n|^u (nN+r)^{\lambda-u}} H_{\lambda} \left(u, v; \frac{r}{nN+r} \right) + \right. \\ &\left. + \cos \pi v \sum_{n \geq (1+r)/N} \frac{\tau_u(n)\tau_v(nN-r)}{n^u (nN-r)^{\lambda-u}} H_{\lambda} \left(u, v; \frac{-r}{nN-r} \right) \right). \end{aligned}$$

Proof. The proof is analogous to lemma 3.6 of [6]. \square

Let

$$(6.15) \quad V_N(l) = \frac{2(-1)^k}{l^{1/2-v}} \sum_{d|l} \sum_{r|d} \mu(d/r) (V_{N,1}(r) + V_{N,2}(r) + V_{N,3}(r)),$$

where

(6.16)

$$V_{N,1}(r) = \frac{\cos \pi v}{r^{-k+v}} \sum_{n \geq (1+r)/N} \frac{\tau_0(n) \tau_v(nN-r)}{(nN-r)^k} H_k \left(0, v; \frac{-r}{nN-r} \right),$$

(6.17)

$$V_{N,2}(r) = \frac{(-1)^k}{r^{-k+v}} \sum_{(1-r)/N \leq n \leq -1} \frac{\tau_0(|n|) \tau_v(nN+r)}{(nN+r)^k} H_k \left(0, v; \frac{r}{nN+r} \right),$$

$$(6.18) \quad V_{N,3}(r) = \frac{(-1)^k}{r^{-k+v}} \sum_{n \geq 1} \frac{\tau_0(n) \tau_v(nN+r)}{(nN+r)^k} H_k \left(0, v; \frac{r}{nN+r} \right).$$

Lemma 6.9. *Assume that $\Re v = 0$ and $(l, p) = 1$. Then*

(6.19)

$$S_1(l, 0, v; N) = \frac{\sigma_{-2v}(l)}{l^{1/2-v}} (\log N + 2(\gamma - \log 2\pi) + \psi(k+v) + \psi(k-v)) - \\ - \frac{1}{l^{1/2-v}} \sum_{d|l} \sum_{r|d} \mu \left(\frac{d}{r} \right) \tau_v(r) r^{-v} \log r + V_N(l)$$

and

$$(6.20) \quad S_2(l, 0, v; N) = \frac{1}{p} \frac{\sigma_{-2v}(l)}{l^{1/2-v}} \times \\ \times (\log(N/p) + 2\gamma - 2\log 2\pi + \psi(k+v) + \psi(k-v)) - \\ - \frac{1}{pl^{1/2-v}} \sum_{d|l} \sum_{r|d} \mu \left(\frac{d}{r} \right) \tau_v(r) r^{-v} \log r + \frac{1}{p} V_{N/p}(l).$$

Proof. We substitute (6.14) into (6.9) and continue $S_1(l, u, v; N)$ analytically to the point $u = 0$. Note that the evaluation of $H_\lambda(u, v; z)$ for $z > 1$ is not straightforward. First, we use the representation of $H_\lambda(u, v; z)$ (for $\Re u > 0$) given by lemma 5.7. But it follows from theorem 4.4 that the integral on the right-hand side of (5.9) converges at $u = 0$ and thus we can eventually let $u = 0$. This allows using (5.9) with $u = 0$ in order to compute $H_\lambda(0, v; z)$ for $z > 1$.

Since $k \geq 2$ all the series converge and, therefore, we can let $\lambda = k$. In order to show that $S_1(l, u, v; N)$ doesn't have a pole at $u = 0$ we

consider its non-analytic summands

$$\begin{aligned} \left(\frac{d}{l}\right)^u & \left(\frac{\zeta(1+2u)}{d^{1/2+u+v}} + \left(\frac{4\pi^2}{N}\right)^{2u} \frac{\Gamma(k-u+v)\Gamma(k-u-v)}{\Gamma(k+u+v)\Gamma(k+u-v)} \times \right. \\ & \left. \times \zeta(1-2u) \sum_{r|d} \frac{\mu(d/r)}{d^{1/2+u-v}} \frac{\tau_v(r)}{r^{v-2u}} \right). \end{aligned}$$

By Möbius inversion

$$\sum_{r|d} \mu(r) \sigma_{-2v}(d/r) = d^{-2v}.$$

This implies that

$$d^{-1/2-v} = \sum_{r|d} \frac{\mu(d/r)}{d^{1/2-v}} \frac{\tau_v(r)}{r^v}$$

and the singularity at $u = 0$ gets canceled. Computing the limit of $S_1(l, u, v; N)$ as $u \rightarrow 0$ gives (6.19). The equality (6.20) can be established similarly. \square

Using lemmas 6.1, 6.2, 6.3 and 6.9 we obtain

Theorem 6.10. *For $\Re v = 0$*

$$\begin{aligned} (6.21) \quad M_2(l, 0, v) &= \left(\frac{\phi(N)}{N} \right)^2 \frac{\sigma_{-2v}(l)}{l^{1/2-v}} \times \\ & \times \left(\log N + 2\gamma - 2 \log 2\pi + 2 \frac{\log p}{p-1} + \psi(k+v) + \psi(k-v) \right) - \\ & - \left(\frac{\phi(N)}{N} \right)^2 \frac{1}{l^{1/2-v}} \sum_{d|l} \sum_{r|d} \mu(d/r) \tau_v(r) r^{-v} \log r + \\ & + O \left(V_N(l) + \frac{1}{p} V_{N/p}(l) + \frac{1}{p^2} V_{N/p^2}(l) \right). \end{aligned}$$

7. THE SECOND MOMENT: $\nu = 2$

In this section we consider the most difficult case $N = p^2$.

Lemma 7.1. *Suppose that $\Re u > 1/2$, $\Re v = 0$, $k \geq 2$ and $N = p^2$ with p prime. Then*

$$(7.1) \quad M_2(l, u, v) = S(l, u, v; p^2) - \frac{1}{p - p^{-1}} S(l, u, v; p),$$

where $S(l, u, v; N)$ is defined by (6.2).

Using lemmas 6.2, 6.3 and 6.9 we obtain that $S(l, 0, v; p^2)$ satisfies the following asymptotic formula with $N = p^2$

$$(7.2) \quad S(l, 0, v; p^2) = \frac{\phi(N)}{N} \frac{\sigma_{-2v}(l)}{l^{1/2-v}} \times \\ \times \left(\log N + 2\gamma - 2\log 2\pi + \frac{\log p}{p-1} + \psi(k+v) + \psi(k-v) \right) - \\ - \frac{\phi(N)}{N} \frac{1}{l^{1/2-v}} \sum_{d|l} \sum_{r|d} \mu(d/r) \tau_v(r) r^{-v} \log r + \\ + O\left(V_N(l) + \frac{1}{p} V_{N/p}(l)\right).$$

The sum $S(l, u, v; p)$ is more involved because in that case we cannot apply the property of vanishing of Kloosterman sums given by lemma 2.1.

Lemma 7.2. *One has*

$$(7.3) \quad S(l, u, v; p) = S_1 - S_2 - S_3 + S_4,$$

where $S_i = S_i(l, u, v; p)$ satisfy (6.6), $T_i(l, u, v; p)$ are defined by (6.3), (6.4) for $i = 1, 2$ and

$$(7.4) \quad T_3(l, u, v; p) = \sum_{m,n=1}^{\infty} \frac{1}{(pmn)^{1/2+u}} \left(\frac{m}{pn}\right)^v \Delta_{2k,p}(md, pn),$$

$$(7.5) \quad T_4(l, u, v; p) = \frac{1}{p^{1+2u}} \sum_{m,n=1}^{\infty} \frac{1}{(mn)^{1/2+u}} \left(\frac{m}{n}\right)^v \Delta_{2k,p}(mdp, np).$$

The asymptotic formula for $S_1(l, 0, v; p)$ is given by equation (6.19). In the following subsections we evaluate S_2 , S_3 and S_4 . The main difference with the previous section is that for $N = p$ the asymptotic formulas for S_i , $i = 2, 3, 4$ will contain extra summands coming from poles of the Lerch zeta function, namely

$$(7.6) \quad E_p^2(u, v, \lambda) = \frac{1}{p} \sum_{q|d} \frac{1}{q} \left(\frac{q}{2\pi\sqrt{d/p}} \right)^{1-2u+2v} \times \\ \times \sum_{\substack{b=1 \\ (b, qp)=1}}^{qp} \zeta(0, b/qp; 1+2v) \frac{\Gamma(\lambda - u + v)}{\Gamma(\lambda + u - v)},$$

$$(7.7) \quad E_p^3(u, v, \lambda) = \frac{p^{2v} - 1}{p} \frac{\zeta(1 - 2v)}{(2\pi\sqrt{d/p})^{1-2u-2v}} \frac{\Gamma(\lambda - u - v)}{\Gamma(\lambda + u + v)},$$

$$(7.8) \quad E_p^4(u, v, \lambda) = \frac{1}{p} \sum_{\substack{q|d \\ q \neq 1}} \frac{1}{q} \left(\frac{q}{2\pi\sqrt{d}} \right)^{1-2u+2v} \times \\ \times \sum_{\substack{b=1 \\ (b, qp)=1}}^{qp} \zeta(0, b/q, 1 + 2v) \frac{\Gamma(\lambda - u + v)}{\Gamma(\lambda + u - v)} + \frac{p-1}{p} \times \\ \times \left(\frac{\zeta(1 + 2v)}{(2\pi\sqrt{d})^{1-2u+2v}} \frac{\Gamma(\lambda - u + v)}{\Gamma(\lambda + u - v)} + \frac{\zeta(1 - 2v)}{(2\pi\sqrt{d})^{1-2u-2v}} \frac{\Gamma(\lambda - u - v)}{\Gamma(\lambda + u + v)} \right).$$

The asymptotics of S_2 and S_3 contains also an error term of the following shape

$$(7.9) \quad W_p(l) = \frac{2i^{2k}}{l^{1/2-v}} \sum_{d|l} \sum_{r|d} \mu(d/r) r^{\lambda-v} \times \\ \left(\cos \pi \lambda \sum_{a=1}^{\infty} \frac{\tau_v(a) \tau_0(|pa - r|)}{(ap)^\lambda} H_\lambda \left(0, v, \frac{r}{ap} \right) + \right. \\ \left. + \cos \pi v \sum_{a=1}^{\infty} \frac{\tau_v(a) \tau_0(pa + r)}{(ap)^\lambda} H_\lambda \left(0, v, -\frac{r}{ap} \right) \right).$$

Evaluating S_2 , S_3 and S_4 we show that

$$S(l, 0, v, p) = -\frac{1 - 1/p}{l^{1/2-v}} \sum_{d|l} \sum_{r|d} \mu(d/r) \tau_v(r) r^{-v} \log r + \frac{\sigma_{-2v}(l)}{l^{1/2-v}} \times \\ \times [(1 + 1/p^2) \log p + (1 - 1/p)(2\gamma - 2 \log 2\pi + \psi(k + v) + \psi(k - v))] + \\ + 2\pi i^{2k} \sum_{d|l} \left(\frac{d}{l} \right)^{1/2-v} \left[-\frac{E_p^2(0, v, \lambda)}{p^{1/2-v}} - \frac{E_p^3(0, v, \lambda)}{p^{1/2+v}} + \frac{E_p^4(0, v, \lambda)}{p} \right] + \\ + \left(1 + \frac{1}{p^2} \right) V_p(l) - \frac{p+1}{p^2} V_1(l) - \left(\frac{1}{p^{1+v}} + \frac{1}{p^{1-v}} \right) W_p(l).$$

Then lemma 7.1 and equation (7.2) imply the main theorem.

Theorem 7.3. *Let $\Re v = 0$, $k \geq 2$, $N = p^2$ with p prime. If $p|l$ then $M_2(l, 0, v) = 0$. Otherwise*

$$\begin{aligned}
 (7.10) \quad M_2(l, 0, v) = & \frac{\sigma_{-2v}(l)}{l^{1/2-v}} \left[2 \left(1 - \frac{p}{p^2-1} \right) \log p + \right. \\
 & + \left(1 - \frac{1}{p} - \frac{1}{p+1} \right) (2\gamma - 2 \log 2\pi + \psi(k+v) + \psi(k-v)) \Big] - \\
 & - \frac{1 - 1/p - 1/(p+1)}{l^{1/2-v}} \sum_{d|l} \sum_{r|d} \mu(d/r) \tau_v(r) r^{-v} \log r + \\
 & + \frac{2\pi i^{2k}}{p-p^{-1}} \sum_{d|l} \left(\frac{d}{l} \right)^{1/2-v} \left(-\frac{E_p^2(0, v, \lambda)}{p^{1/2-v}} - \frac{E_p^3(0, v, \lambda)}{p^{1/2+v}} + \frac{E_p^4(0, v, l)}{p} \right) + \\
 & + O \left(V_{p^2}(l) + \frac{1}{p} V_p(l) + \frac{1}{p^2} V_1(l) + \frac{1}{p^2} W_p(l) \right).
 \end{aligned}$$

7.1. The sum S_4 . In this subsection we evaluate $S_4(u, v, \lambda; p)$ as $u \rightarrow 0$.

Theorem 7.4. *For $\Re v = 0$*

$$\begin{aligned}
 S_4(0, v, \lambda; p) = & -\frac{1}{pl^{1/2-v}} \sum_{d|l} \sum_{r|d} \mu(d/r) \tau_v(r) r^{-v} \log r + \\
 & + \frac{\sigma_{-2v}(l)}{pl^{1/2-v}} \left(\frac{\log p}{p} + 2\gamma - 2 \log 2\pi + \psi(k+v) + \psi(k-v) \right) + \\
 & + \frac{p-1}{p^2} V_1(l) + \frac{1}{p^2} V_p(l) + \frac{2\pi i^{2k}}{p} \sum_{d|l} \left(\frac{d}{l} \right)^{1/2-v} E_p^4(0, v; \lambda).
 \end{aligned}$$

According to the trace formula (3.2)

$$(7.11) \quad T_4(l, u, v; p) = \frac{1}{p^{1+2u}} \left(\frac{\zeta(1+2u)}{d^{1/2+u}} + 2\pi i^{2k} D_p^4(u, v; \lambda) \right),$$

where

$$\begin{aligned}
 (7.12) \quad D_p^4(u, v; \lambda) = & \frac{1}{p} \sum_{m,n=1}^{\infty} \frac{(m/n)^v}{(mn)^{1/2+u}} \times \\
 & \times \sum_{q=1}^{\infty} \frac{Kl(dpm, pn; qp)}{q} J_{2\lambda-1} \left(4\pi \frac{\sqrt{dmn}}{q} \right).
 \end{aligned}$$

For $\Re s > 1 + |\Re v|$ let

$$(7.13) \quad G_4^*(s, v; d, q) = \sum_{m,n=1}^{\infty} \frac{Kl(dpm, pn; qp)}{(mn)^s} \left(\frac{m}{n}\right)^v = \\ = \sum_{a,b=1}^{qp} \delta_{qp}(ab-1) \zeta(0, ad/q, s-v) \zeta(0, b/q, s+v).$$

In particular, if $q|d$

$$(7.14) \quad G_4^*(s, v; d, q) = \zeta(s-v) \sum_{\substack{b=1 \\ (b, qp)=1}}^{qp} \zeta(0, b/q, s+v),$$

$$(7.15) \quad G_4^*(s, v; d, 1) = \phi(p) \zeta(s-v) \zeta(s+v).$$

Note that $\zeta(0, ad/q, s-v)$ has a pole when $q|d$. Since $(b, qp) = 1$ the function $\zeta(0, b/q, s+v)$ has a pole at $q = 1$.

Lemma 7.5. *For $q \in \mathbf{N}$ and $v \in \mathbf{C}$ the function $G_4^*(s, v; d, q)$ can be meromorphically continued on the whole complex plane as a function of complex variable s . Furthermore, for $\Re s < -|\Re v|$ one has*

$$(7.16) \quad G_4^*(s, v; d, q) = 2\Gamma(1-s+v)\Gamma(1-s-v) \left(\frac{2\pi}{q}\right)^{2s-2} \times \\ \times (p-1 + \delta_p(q)) \left(-\cos \pi s \sum_{\substack{m,n=1 \\ (n,q)=1}}^{\infty} \frac{\delta_q(mn-d)}{(mn)^{1-s}} \left(\frac{m}{n}\right)^{-v} + \right. \\ \left. + \cos \pi v \sum_{\substack{m,n=1 \\ (n,q)=1}}^{\infty} \frac{\delta_q(mn+d)}{(mn)^{1-s}} \left(\frac{m}{n}\right)^{-v} \right).$$

Proof. Following the arguments of lemma 2.2 of [6] we obtain

$$\sum_{a,b=1}^{qp} \delta_{qp}(ab-1) \zeta(ad/q, 0, 1-s+v) \zeta(b/q, 0, 1-s-v) = \\ = q^{2-2s} \sum_{m,n=1}^{\infty} \frac{1}{(mn)^{1-s}} \left(\frac{n}{m}\right)^v \sum_{\substack{a,b=1 \\ b \equiv n \pmod{q} \\ ad \equiv m \pmod{q}}}^{qp} \delta_{qp}(ab-1).$$

We proceed to evaluate the last sum. There exists $j \pmod{p}$ such that $b = n + jq$. Since $ab \equiv 1 \pmod{qp}$, we have $an + ajq \equiv 1 \pmod{qp}$. The requirements $(a, qp) = 1$, $aj + \frac{an-1}{q} \equiv 0 \pmod{p}$, $an \equiv 1 \pmod{q}$ imply that j is unique. We are left to compute the number of solutions of the system

$$\begin{cases} (a, qp) = 1 \\ an \equiv 1 \pmod{q} \\ m \equiv ad \pmod{q} \end{cases} \Leftrightarrow \begin{cases} (a, qp) = 1 \\ a \equiv n^{-1} \pmod{q} \\ (n, q) = 1 \\ m \equiv ad \pmod{q} \end{cases} \Leftrightarrow \begin{cases} (n, q) = 1 \\ mn \equiv d \pmod{q} \\ (n^{-1} + wq, qp) = 1 \end{cases}$$

for $w \pmod{p}$. Since $(n^{-1}, q) = 1$, we have $(n^{-1} + wq, p) = 1$. The number of w such that $n^{-1} + wq \equiv 0 \pmod{p}$ is equal to one if $(q, p) = 1$ and is equal to zero otherwise. Therefore,

$$\sum_{\substack{a, b=1 \\ b \equiv n \pmod{q} \\ ad \equiv m \pmod{q}}}^{qp} \delta_{qp}(ab - 1) = (p - 1 + \delta_p(q)) \mathbf{1}_{(n, q)=1} \delta_q(mn - d).$$

The rest of the proof is analogous to lemma 2.2 of [6]. □

Next, we apply functional equation (7.16) to evaluate (7.12).

Lemma 7.6. *Let $\Re \lambda - 1 > \Re u > 3/4$ and $\Re v = 0$. For $1 - 2\Re \lambda < \Delta < -1 - 2\Re u$ one has*

$$\begin{aligned} (7.17) \quad D_p^4(u, v; \lambda) &= \frac{(2\pi)^{2u-1}}{2\pi i} \int_{\Re s = \Delta} \Gamma(u, v, \lambda; s) \times \\ &\times \left(\sin \pi(u + s/2) \sum_{q=1}^{\infty} \sum_{\substack{m, n=1 \\ (n, q)=1}}^{\infty} \frac{\delta_q(mn - d)}{(mn)^{1/2-u-s/2}} \left(\frac{m}{n}\right)^{-v} \frac{p-1+\delta_p(q)}{pq^{2u}} + \right. \\ &\left. + \cos \pi v \sum_{q=1}^{\infty} \sum_{\substack{m, n=1 \\ (n, q)=1}}^{\infty} \frac{\delta_q(mn + d)}{(mn)^{1/2-u-s/2}} \left(\frac{m}{n}\right)^{-v} \frac{p-1+\delta_p(q)}{pq^{2u}} \right) \frac{ds}{d^{s/2}} + E_p^4(u, v; \lambda), \end{aligned}$$

where the last summand, representing the contribution of poles, is defined by (7.8).

Lemma 7.6 implies that

$$(7.18) \quad D_p^4(u, v; \lambda) = \frac{p-1}{p} D_1^*(u, v; \lambda) + \frac{1}{p} D_p^*(u, v; \lambda) + E_p^4(u, v; \lambda),$$

where the function $D_N^*(u, v; \lambda)$, defined by (6.8), satisfies (6.14). Using (7.18), (6.14) and letting $u \rightarrow 0$, we compute the asymptotics of $S_4(u, v, \lambda; p)$. Theorem 7.4 follows.

7.2. The sum S_3 . The main result of this subsection is the following theorem.

Theorem 7.7. *For $\Re v = 0$*

$$\begin{aligned} S_3(l, 0, \lambda; p) = & -\frac{1}{pl^{1/2-v}} \sum_{d|l} \sum_{r|d} \mu(d/r) \tau_v(r) r^{-v} \log r + \\ & + \frac{\sigma_{-2v}(l)}{pl^{1/2-v}} (2\gamma - 2 \log 2\pi + \psi(k+v) + \psi(k-v)) + \\ & + \frac{1}{p} V_1(l) + \frac{1}{p^2} V_p(l) + \frac{1}{p^{1+v}} W_p(l) + \frac{2\pi i^{2k}}{p} \sum_{d|l} \left(\frac{d}{l}\right)^{1/2-v} E_p^3(0, v; \lambda). \end{aligned}$$

According to the trace formula (3.2)

$$\begin{aligned} (7.19) \quad S_3(l, u, v; p) = & \sum_{d|l} \left(\frac{d}{l}\right)^{1/2-v} \frac{1}{p^{1/2+v}} \left(\frac{d}{pl}\right)^u \times \\ & \times \left(\frac{\zeta(1+2u)}{p^{1/2+u-v} d^{1/2+u+v}} + 2\pi i^{2k} D_p^3(u, v; \lambda) \right), \end{aligned}$$

where

$$\begin{aligned} (7.20) \quad D_p^3(u, v; \lambda) = & \frac{1}{p} \sum_{m,n=1}^{\infty} \frac{(m/n)^v}{(mn)^{1/2+u}} \times \\ & \times \sum_{q=1}^{\infty} \frac{Kl(dm, pn; qp)}{q} J_{2\lambda-1} \left(4\pi \frac{\sqrt{dmn/p}}{q} \right). \end{aligned}$$

For $\Re s > 1 + |\Re v|$ let

$$\begin{aligned} (7.21) \quad G_3^*(s, v; d, q) = & \sum_{m,n=1}^{\infty} \frac{Kl(dm, pn; qp)}{(mn)^s} \left(\frac{m}{n}\right)^v = \\ = & \sum_{a,b=1}^{qp} \delta_{qp}(ab-1) \zeta(0, ad/qp, s-v) \zeta(0, b/q, s+v). \end{aligned}$$

Since $(d, p) = 1$, $(a, qp) = 1$, the function $\zeta(0, ad/qp, s-v)$ has no poles. The second function $\zeta(0, b/q, s+v)$ has a pole if $q = 1$. Namely,

$$(7.22) \quad G_3^*(s, v; d, 1) = (p^{1-s+v} - 1) \zeta(s-v) \zeta(s+v)$$

has a pole at $s = 1 + v$.

Lemma 7.8. *For $q \in \mathbf{N}$ and $v \in \mathbf{C}$ the function $G_3^*(s, v; d, q)$ can be meromorphically continued on the whole complex plane as a function of complex variable s . Furthermore, for $\Re s < -|\Re v|$ one has*

$$(7.23) \quad G_3^*(s, v; d, q) = 2\Gamma(1 - s + v)\Gamma(1 - s - v) \left(\frac{2\pi}{q}\right)^{2s-2} p^{1-s+v} \times \\ \times \left(-\cos \pi s \sum_{\substack{m,n=1 \\ (n,q)=1 \\ (m,p)=1}}^{\infty} \frac{\delta_q(mn - d)}{(mn)^{1-s}} \left(\frac{m}{n}\right)^{-v} + \right. \\ \left. + \cos \pi v \sum_{\substack{m,n=1 \\ (n,q)=1 \\ (m,p)=1}}^{\infty} \frac{\delta_q(mn + d)}{(mn)^{1-s}} \left(\frac{m}{n}\right)^{-v} \right).$$

Proof. Following the proof of lemma 2.2 of [6] we need to calculate the sum

$$\sum_{q,b=1}^{qp} \delta_{qp}(ab - 1) \zeta(ad/(qp), 0, 1 - s + v) \zeta(b/q, 0, 1 - s - v) = \\ = q^{2-2s} p^{1-s+v} \sum_{m,n=1}^{\infty} \frac{(n/m)^v}{(mn)^{1-s}} \sum_{\substack{a,b=1 \\ ad \equiv m \pmod{qp} \\ b \equiv n \pmod{q}}}^{qp} \delta_{qp}(ab - 1).$$

The last sum can be evaluated by computing the number of solution a, b of the following system

$$\begin{cases} ab \equiv 1 \pmod{qp} \\ ad \equiv m \pmod{qp} \\ b = n + wq, \quad w \pmod{p} \end{cases} \Leftrightarrow \begin{cases} an + awq \equiv 1 \pmod{qp} \\ (a, p) = 1 \\ ad \equiv m \pmod{qp} \end{cases}.$$

Since $aw + \frac{an-1}{q} \equiv 0 \pmod{p}$, the value of w is unique. Hence

$$\begin{cases} an \equiv 1 \pmod{q} \\ ad \equiv m \pmod{qp} \\ (a, p) = 1 \end{cases} \Leftrightarrow \begin{cases} (a, p) = 1 \\ (n, q) = 1 \\ ad \equiv m \pmod{qp} \\ a = n^{-1} + jq, \quad j \pmod{p} \end{cases}.$$

This gives $(n, q) = 1$, $(n^{-1} + jq, p) = 1$, $dn^{-1} + djq \equiv m \pmod{qp}$. Therefore, $dj + \frac{dn^{-1}-m}{q} \equiv 0 \pmod{p}$. Since $(d, p) = 1$, we have $j =$

$\frac{m-dn^{-1}}{q}d^* \pmod{p}$, where $nn^{-1} \equiv 1 \pmod{q}$, $dd^* \equiv 1 \pmod{p}$. Next, we compute the number of $j_0 = \frac{m-dn^{-1}}{q}d^* \pmod{p}$ for which the condition $(n^{-1}+j_0q, p) = 1$ is satisfied. If $n^{-1}+j_0q \equiv 0 \pmod{p}$, then $n^{-1}+(m-dn^{-1})d^* \equiv 0 \pmod{p}$. Thus $md^* \equiv 0 \pmod{p}$ and, since $(d, p) = 1$, we have $m \equiv 0 \pmod{p}$. Hence there is no solutions if $m \equiv 0 \pmod{p}$ and there is one solution j_0 if $(m, p) = 1$. The final set of conditions is the following: $(n, q) = 1$, $(m, p) = 1$, $mn \equiv d \pmod{q}$. This implies the statement of the lemma. \square

Next, we apply functional equation (7.23) to evaluate (7.20).

Lemma 7.9. *Let $\Re\lambda - 1 > \Re u > 3/4$ and $\Re v = 0$. For $1 - 2\Re\lambda < \Delta < -1 - 2\Re u$ one has*

$$(7.24) \quad D_p^3(u, v; \lambda) = E_p^3(u, v; \lambda) + \frac{(2\pi)^{2u-1}}{2\pi i} \int_{\Re s = \Delta} \Gamma(u, v, \lambda; s) \times \\ \times \left(\sin \pi(u + s/2) \sum_{q=1}^{\infty} \sum_{\substack{m, n=1 \\ (m, p)=1 \\ (n, q)=1}}^{\infty} \frac{\delta_q(mn - d)}{q^{2u}(mn)^{1/2-u-s/2}} \left(\frac{m}{n}\right)^{-v} + \right. \\ \left. + \cos \pi v \sum_{q=1}^{\infty} \sum_{\substack{m, n=1 \\ (m, p)=1 \\ (n, q)=1}}^{\infty} \frac{\delta_q(mn + d)}{q^{2u}(mn)^{1/2-u-s/2}} \left(\frac{m}{n}\right)^{-v} \right) \frac{p^{1/2-u-s/2+v} ds}{p(\sqrt{d/p})^s}.$$

Lemma 7.10. *Let $\Re\lambda - 1 > \Re u > 3/4$ and $\Re v = 0$. For $1 - 2\Re\lambda < \Delta < -1 - 2\Re u$ one has*

$$(7.25) \quad D_p^3(u, v; \lambda) = \frac{1}{p^{1/2+u-v}} D_1^*(u, v, \lambda) + E_p^3(u, v; \lambda) + \\ + \frac{(2\pi)^{2u-1}}{2\pi i} \int_{\Re s = \Delta} \Gamma(u, v, \lambda; s) \times \\ \times \left(\sin \pi(u + s/2) \sum_{r|d} \frac{\mu(d/r)}{(d/r)^{1/2+u-v-s/2}} \sum_{a=1}^{\infty} \frac{\tau_v(a)\tau_u(|pa-r|)}{a^{1/2-u-s/2}|pa-r|^u} + \right. \\ \left. + \cos \pi v \sum_{r|d} \frac{\mu(d/r)}{(d/r)^{1/2+u-v-s/2}} \sum_{a=1}^{\infty} \frac{\tau_v(a)\tau_u(pa+r)}{a^{1/2-u-s/2}(pa+r)^u} \right) \frac{ds}{p(d/p)^{s/2}}.$$

Proof. Consider the following decomposition

$$\sum_{q=1}^{\infty} \sum_{\substack{m,n=1 \\ (m,p)=(n,q)=1}}^{\infty} \frac{\delta_q(mn-d)}{q^{2u}(mn)^z} \left(\frac{m}{n}\right)^{-v} := A_1 - A_2,$$

where

$$A_1 = \sum_{q=1}^{\infty} \sum_{\substack{m,n=1 \\ (n,q)=1}}^{\infty} \frac{\delta_q(mn-d)}{q^{2u}(mn)^z} \left(\frac{m}{n}\right)^{-v},$$

$$A_2 = \frac{1}{p^{z+v}} \sum_{q=1}^{\infty} \sum_{\substack{m,n=1 \\ (n,q)=1}}^{\infty} \frac{\delta_q(pmn-d)}{q^{2u}(mn)^z} \left(\frac{m}{n}\right)^{-v}.$$

The sum A_1 gives the first term in (7.25). By Möbius inversion

$$A_2 = \frac{1}{p^{z+v}} \sum_{r|d} \frac{\mu(d/r)}{(d/r)^{z+2u-v}} \sum_{q=1}^{\infty} \frac{1}{q^{2u}} \sum_{m,n=1}^{\infty} \frac{\delta_q(pmn-r)}{(mn)^z} \left(\frac{m}{n}\right)^{-v}.$$

Since $r|d$ we have $(r,p)=1$ and, therefore, there is no m, n such that $pmn=r$. Thus

$$\begin{aligned} \sum_{q=1}^{\infty} \frac{1}{q^{2u}} \sum_{m,n=1}^{\infty} \frac{\delta_q(pmn-r)}{(mn)^z} \left(\frac{m}{n}\right)^{-v} &= \sum_{q=1}^{\infty} \frac{1}{q^{2u}} \sum_{a=1}^{\infty} \frac{\delta_q(pa-r)}{a^z} \tau_v(a) = \\ &= \sum_{a=1}^{\infty} \frac{\tau_v(a)}{a^z} \sigma_{-2u}(|pa-r|) = \sum_{a=1}^{\infty} \frac{\tau_v(a)\tau_u(|pa-r|)}{a^z |pa-r|^u} \end{aligned}$$

and

$$A_2 = \frac{1}{p^{z+v}} \sum_{r|d} \frac{\mu(d/r)}{(d/r)^{z+2u-v}} \sum_{a=1}^{\infty} \frac{\tau_v(a)\tau_u(|pa-r|)}{a^z |pa-r|^u}.$$

□

Integrating over s we obtain

$$(7.26) \quad D_p^3(u, v; \lambda) = \frac{1}{p^{1/2+u-v}} D_1^*(u, v; \lambda) + E_p^3(u, v; \lambda) + \\ + \frac{2(2\pi)^{2u-1}}{p^{\lambda+1/2}} \sum_{r|d} \frac{\mu(d/r) r^{\lambda+u-v}}{d^{1/2+u-v}} \times \\ \times \left(\cos \pi(\lambda - u) \sum_{a=1}^{\infty} \frac{\tau_v(a) \tau_u(|pa - r|)}{a^{\lambda-u} |pa - r|^u} H_{\lambda} \left(u, v, \frac{r}{ap} \right) + \right. \\ \left. + \cos \pi v \sum_{a=1}^{\infty} \frac{\tau_v(a) \tau_u(pa + r)}{a^{\lambda-u} (pa + r)^u} H_{\lambda} \left(u, v, -\frac{r}{ap} \right) \right),$$

where the function $D_1^*(u, v; \lambda)$, defined by (6.8), satisfies (6.14). Letting $u \rightarrow 0$, we prove theorem 7.7.

7.3. The sum S_2 . This subsection is devoted to proving the formula below.

Theorem 7.11. *For $\Re v = 0$*

$$(7.27) \quad S_2(l, 0, \lambda; p) = -\frac{1}{pl^{1/2-v}} \sum_{d|l} \sum_{r|d} \mu(d/r) \tau_v(r) r^{-v} \log r + \\ + \frac{\sigma_{-2v}(l)}{pl^{1/2-v}} (2\gamma - 2 \log 2\pi + \psi(k+v) + \psi(k-v)) + \\ + \frac{1}{p} V_1(l) + \frac{1}{p^{1-v}} W_p(l) + \frac{2\pi i^{2k}}{p^{1/2-v}} \sum_{d|l} \left(\frac{d}{l} \right)^{1/2-v} E_p^2(0, v, \lambda).$$

Applying the trace formula (3.2) we have

$$(7.28) \quad S_2(l, u, v; p) = \sum_{d|l} \left(\frac{d}{l} \right)^{1/2-v} \frac{1}{p^{1/2-v}} \left(\frac{d}{pl} \right)^u \times \\ \times \left(\frac{\zeta(1+2u)}{(dp)^{1/2+u+v}} + 2\pi i^{2k} D_p^2(u, v; \lambda) \right),$$

where

$$(7.29) \quad D_p^2(u, v; \lambda) = \frac{1}{p} \sum_{m,n=1}^{\infty} \frac{(m/n)^v}{(mn)^{1/2+u}} \times \\ \times \sum_{q=1}^{\infty} \frac{Kl(dmp, n; qp)}{q} J_{2\lambda-1} \left(4\pi \frac{\sqrt{dmn/p}}{q} \right).$$

For $\Re s > 1 + |\Re v|$ let

$$(7.30) \quad G_2^*(s, v; d, q) = \sum_{m, n=1}^{\infty} \frac{Kl(dpm, n; qp)}{(mn)^s} \left(\frac{m}{n}\right)^v =$$

$$= \sum_{a, b=1}^{qp} \delta_q(ab-1) \zeta(0, ad/q, s-v) \zeta(0, b/(qp), s+v).$$

Note that $\zeta(0, ad/q; s-v)$ has a pole when $q|d$ and $\zeta(0, b/(qp), s+v)$ does not have poles.

Lemma 7.12. *For $q \in \mathbf{N}$ and $v \in \mathbf{C}$ the function $G_2^*(s, v; d, q)$ can be meromorphically continued on the whole complex plane as a function of complex variable s . Furthermore, for $\Re s < -|\Re v|$ one has*

$$(7.31) \quad G_2^*(s, v; d, q) = 2\Gamma(1-s+v)\Gamma(1-s-v) \left(\frac{2\pi}{q}\right)^{2s-2} p^{1-s-v} \times$$

$$\times \left(-\cos \pi s \sum_{\substack{m, n=1 \\ (n, qp)=1}}^{\infty} \frac{\delta_q(mn-d)}{(mn)^{1-s}} \left(\frac{m}{n}\right)^{-v} + \right.$$

$$\left. + \cos \pi v \sum_{\substack{m, n=1 \\ (n, qp)=1}}^{\infty} \frac{\delta_q(mn+d)}{(mn)^{1-s}} \left(\frac{m}{n}\right)^{-v} \right).$$

Proof. Our arguments follow the proof of lemma 2.2 of [6] with the difference that now we need to compute

$$\sum_{a, b=1}^{qp} \delta_q(ab-1) \zeta(ad/q, 0, 1-s+v) \zeta(b/(qp), 0, 1-s-v) =$$

$$= q^{2-2s} p^{1-s-v} \sum_{m, n=1}^{\infty} \frac{(n/m)^v}{(mn)^{1-s}} \sum_{\substack{a, b=1 \\ ad \equiv m \pmod{q} \\ b \equiv n \pmod{qp}}}^{qp} \delta_{qp}(ab-1).$$

To evaluate the sum over a, b we consider the following system

$$\begin{cases} ab \equiv 1 \pmod{qp} \\ ad \equiv m \pmod{q} \\ b \equiv n \pmod{qp} \end{cases} \Leftrightarrow \begin{cases} an \equiv 1 \pmod{qp} \\ ad \equiv m \pmod{q} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} an \equiv 1 \pmod{qp} \\ adp \equiv mp \pmod{qp} \end{cases} \Leftrightarrow \begin{cases} (n, qp) = 1 \\ dp \equiv mnp \pmod{qp} \end{cases}.$$

This gives the condition $(n, qp) = 1$, $mn \equiv d \pmod{q}$ in the final formula. \square

Next, we apply functional equation (7.31) to evaluate (7.29).

Lemma 7.13. *Let $\Re\lambda - 1 > \Re u > 3/4$ and $\Re v = 0$. For $1 - 2\Re\lambda < \Delta < -1 - 2\Re u$ one has*

$$(7.32) \quad D_p^2(u, v; \lambda) = E_p^2(u, v; \lambda) + \frac{(2\pi)^{2u-1}}{2\pi i} \int_{\Re s = \Delta} \Gamma(u, v, \lambda; s) \times \\ \times \left(\sin \pi(u + s/2) \sum_{q=1}^{\infty} \sum_{\substack{m, n=1 \\ (n, qp)=1}}^{\infty} \frac{\delta_q(mn - d)}{q^{2u}(mn)^{1/2-u-s/2}} \left(\frac{m}{n}\right)^{-v} + \right. \\ \left. + \cos \pi v \sum_{q=1}^{\infty} \sum_{\substack{m, n=1 \\ (n, qp)=1}}^{\infty} \frac{\delta_q(mn + d)}{q^{2u}(mn)^{1/2-u-s/2}} \left(\frac{m}{n}\right)^{-v} \right) \frac{p^{1/2-u-s/2-v} ds}{p(\sqrt{d/p})^s}.$$

Lemma 7.14. *Let $\Re\lambda - 1 > \Re u > 3/4$ and $\Re v = 0$. For $1 - 2\Re\lambda < \Delta < -1 - 2\Re u$ one has*

$$(7.33) \quad D_p^2(u, v; \lambda) = \frac{1}{p^{1/2+u+v}} D_1^*(u, v, \lambda) + E_p^2(u, v; \lambda) + \\ + \frac{(2\pi)^{2u-1}}{2\pi i} \int_{\Re s = \Delta} \Gamma(u, v, \lambda; s) \times \\ \times \left(\sin \pi(u + s/2) \sum_{r|d} \frac{\mu(d/r)}{(d/r)^{1/2+u-v-s/2}} \sum_{a=1}^{\infty} \frac{\tau_v(a) \tau_u(|pa - r|)}{a^{1/2-u-s/2} |pa - r|^u} + \right. \\ \left. + \cos \pi v \sum_{r|d} \frac{\mu(d/r)}{(d/r)^{1/2+u-v-s/2}} \sum_{a=1}^{\infty} \frac{\tau_v(a) \tau_u(pa + r)}{a^{1/2-u-s/2} (pa + r)^u} \right) \frac{ds}{p(d/p)^{s/2}}.$$

Proof. Consider

$$\sum_{q=1}^{\infty} \sum_{\substack{m, n=1 \\ (n, qp)=1}}^{\infty} \frac{\delta_q(mn - d)}{q^{2u}(mn)^z} \left(\frac{m}{n}\right)^{-v} = \sum_{q=1}^{\infty} \sum_{\substack{m, n=1 \\ (n, q)=1}}^{\infty} \frac{\delta_q(mn - d)}{q^{2u}(mn)^z} \left(\frac{m}{n}\right)^{-v} - \\ - \frac{1}{p^{z-v}} \sum_{q=1}^{\infty} \sum_{\substack{m, n=1 \\ (pn, q)=1}}^{\infty} \frac{\delta_q(pmn - d)}{q^{2u}(mn)^z} \left(\frac{m}{n}\right)^{-v}.$$

Since $pmn \equiv d \pmod{q}$ and $(d, p) = 1$, one has $(q, p) = 1$ and the condition $(pn, q) = 1$ in the second sum can be replaced by $(n, q) = 1$. The rest of the proof is similar to lemma 7.10. \square

Integrating over s and letting $u \rightarrow 0$ we prove theorem 7.11.

8. ERROR TERMS

In this section we estimate the error terms defined by (6.15), (6.16), (6.17), (6.18), (7.6), (7.7), (7.8) and (7.9). Let

$$v := it, \quad t \in \mathbf{R}, \quad T := 3 + |t|, \quad \lambda := k,$$

$$n_0 := \left\lceil \frac{r+1}{N} \right\rceil + 1, \quad n_1 := 5n_0, \quad n_2 := \frac{r}{N}(1 + T^2).$$

Lemma 8.1. *One has*

$$(8.1) \quad V_{N,1}(r) \ll \begin{cases} \frac{r}{N}(rkT^2)^\epsilon(k+T) & r \geq N/(1+T^2); \\ \left(\frac{rT^2}{4(N-r)}\right)^k \frac{N^\epsilon}{T} & r < N/(1+T^2). \end{cases}$$

Proof. Estimating (6.16) we obtain

$$(8.2) \quad V_{N,1}(r) \ll \cosh \pi t \sum_{n \geq n_0} \frac{n^\epsilon(nN-r)^\epsilon r^k}{(nN-r)^k} H_k \left(0, v; \frac{-r}{nN-r} \right).$$

- (1) Assume that $r \geq N/2$. We split the sum over n in (8.2) into three parts: $n_0 \leq n < n_1$, $n_1 \leq n < n_2$ and $n \geq n_2$. Estimate (5.5) implies that the first sum is bounded by

$$\begin{aligned} \sum_{n_0 \leq n < n_1} (nN)^\epsilon \frac{1}{\sqrt{T}} \left[(rk)^\epsilon + \frac{k+T}{\sqrt{T}} \sqrt{\frac{nN-r}{r}} \right] &\ll \\ &\ll \frac{(rT)^\epsilon}{\sqrt{T}} \left[n_1 k^\epsilon + \frac{k+T}{\sqrt{T}} \frac{r}{N} \sqrt{\frac{n_1 N - r}{r}} \right] \ll (rkT)^\epsilon \frac{r}{N} \frac{k+T}{T}. \end{aligned}$$

Inequality (5.4) gives the following bound for the second sum

$$\begin{aligned} \sum_{n_1 \leq n < n_2} (nN)^\epsilon \frac{\sqrt{r}}{\sqrt{nN-r}} &\ll (rT)^\epsilon r^{1/2} \times \\ &\times \left(\frac{1}{(n_1 N - r)^{1/2}} + \frac{(n_2 N - r)^{1/2}}{N} \right) \ll (Tr)^\epsilon \frac{rT}{N}. \end{aligned}$$

Applying (5.3) we estimate the third sum

$$\begin{aligned}
 (8.3) \quad & \sum_{n \geq n_2} (nN - r)^\epsilon \frac{r^k}{(nN - r)^k} \frac{T^{2k-1}}{4^k \sqrt{k}} \left[1 + \frac{k+T}{\sqrt{k}} \sqrt{\frac{r}{nN - r}} \right] \ll \\
 & \ll \frac{r^k T^{2k-1}}{4^k \sqrt{k}} \left[\frac{(n_2 N - r)^\epsilon}{N(n_2 N - r)^{k-1}} + \frac{k+r}{\sqrt{k}} \frac{\sqrt{r}(n_2 N - r)^\epsilon}{N(n_2 N - r)^{k-1/2}} \right] \ll \\
 & \ll (rT^2)^\epsilon \frac{rT}{4^k N}.
 \end{aligned}$$

Combining the last three bounds we have

$$V_{N,1}(r) \ll \frac{r}{N} (rkT^2)^\epsilon (k+T) \quad \text{for } r \geq N/2.$$

- (2) Assume that $N/(1+T^2) \leq r < N/2$. We split the sum over n in (8.2) into two parts: $n_0 \leq n < n_2$ and $n \geq n_2$. Estimate (5.4) implies that the first sum is bounded by

$$\begin{aligned}
 \sum_{n_0 \leq n < n_2} (nN)^\epsilon \frac{\sqrt{r}}{\sqrt{nN - r}} & \ll (rT)^\epsilon r^{1/2} \times \\
 & \times \left(\frac{1}{(n_0 N - r)^{1/2}} + \frac{(n_2 N - r)^{1/2}}{N} \right) \ll (Tr)^\epsilon \frac{rT}{N}.
 \end{aligned}$$

It follows from the last estimate and (8.3) that

$$V_{N,1}(r) \ll \frac{r}{N} (rkT^2)^\epsilon (k+T) \quad \text{for } N/(1+T^2) \leq r < N/2.$$

- (3) Assume that $r < N/(1+T^2)$. Applying (5.3) we obtain (in the same way as in (8.3))

$$V_{N,1}(r) \ll \left(\frac{rT^2}{4(N-r)} \right)^k \frac{N^\epsilon}{T} \quad \text{for } r < N/(1+T^2).$$

□

Lemma 8.2. *For any $\epsilon > 0$, $r > N$*

$$(8.4) \quad V_{N,2}(r) \ll_\epsilon \frac{r^{1+\epsilon}}{N}.$$

Proof. By lemma 5.7 we have that

$$\begin{aligned}
 V_{N,2}(r) & \ll \sum_{(1-r)/N \leq n \leq -1} \frac{\tau_0(|n|)\tau_v(nN+r)}{r^v} \int_0^\infty J_{2k-1}(x) \times \\
 & \times k^+ \left(x \sqrt{\frac{nN+r}{r}}, 1/2+v \right) dx \ll \sum_{1 \leq n \leq (r-1)/N} \left| I \left(\frac{r-nN}{r} \right) \right|,
 \end{aligned}$$

where $I(z)$ is defined by (4.1). Then corollary 4.5 gives the desired result. \square

Lemma 8.3. *For any $\epsilon > 0$*

$$(8.5) \quad V_{N,3}(r) \ll \begin{cases} \frac{(rT^2)^\epsilon r}{N} & r \geq N/(1+T^2); \\ \left(\frac{r}{2(N+r)}\right)^k N^\epsilon & r < N/(1+T^2). \end{cases}$$

Proof. The first bound follows from (5.7) and the second one from (5.8). \square

Theorem 8.4. *For any $\epsilon > 0$*

$$(8.6) \quad V_N(l) \ll \begin{cases} (lkT^2)^\epsilon \frac{l^{1/2}(k+T)}{N} & l \geq N/(1+T^2); \\ \left(\frac{lT^2}{4(N-l)}\right)^k \frac{N^\epsilon}{l^{1/2}T} & l < N/(1+T^2). \end{cases}$$

Proof. By lemmas 8.1, 8.2 and 8.3

$$V_N(l) \ll \frac{1}{l^{1/2}} \sum_{r|l} \left(V_{N,1}(r) + V_{N,2}(r) + V_{N,3}(r) \right) \ll \frac{1}{l^{1/2}} \sum_{r|l} V_{N,1}(r).$$

If $l < N/(1+T^2)$ one can apply the second bound in (8.1) to obtain (8.6). Assume $l \geq N/(1+T^2)$. Then for $N^{1-10\epsilon}/(1+T^2) < r < N/(1+T^2)$ we have

$$\left(\frac{rT^2}{4(N-r)}\right)^k \frac{N^\epsilon}{T} \ll \frac{r}{N} (rkT^2)^\epsilon (k+T).$$

Next, we split the sum over r into two parts. The assertion follows by applying estimates (8.1), i.e. we use the first bound for $r > N^{1-10\epsilon}/(1+T^2)$ and the second bound for $r < N^{1-10\epsilon}/(1+T^2)$. \square

Lemma 8.5. *One has*

$$(8.7) \quad \frac{1}{p^2} W_p(l) \ll \frac{1}{p} V_p(l).$$

Proof. Consider

$$W_p(l) = \frac{2(-1)^k}{l^{1/2-v}} \sum_{d|l} \sum_{r|d} \mu(d/r) (W_{p,1}(r) + W_{p,2}(r) + W_{p,3}(r)),$$

where

$$W_{p,1}(r) = \frac{\cos \pi v}{r^{-k+v}} \sum_{a=1}^{\infty} \frac{\tau_v(a) \tau_0(pa+r)}{(ap)^k} H_k \left(0, v; \frac{-r}{ap} \right),$$

$$W_{p,2}(r) = \frac{(-1)^k}{r^{-k+v}} \sum_{a \geq r/p} \frac{\tau_v(a)\tau_0(pa-r)}{(ap)^k} H_k\left(0, v; \frac{r}{ap}\right),$$

$$W_{p,3}(r) = \frac{(-1)^k}{r^{-k+v}} \sum_{a < r/p} \frac{\tau_v(a)\tau_0(|pa-r|)}{(ap)^k} H_k\left(0, v; \frac{r}{ap}\right).$$

Analogously to lemma 8.1 we obtain

$$W_{p,1}(r) \ll \begin{cases} \frac{r}{p}(rkT^2)^\epsilon(k+T), & r \geq p/T^2 \\ \left(\frac{rT^2}{4p}\right)^k \frac{1}{T\sqrt{k}}, & r < p/T^2. \end{cases}$$

Therefore, $W_{p,1}(r) \ll V_{p,1}(r)$. Next, we estimate $W_{p,2}(r)$. It is convenient to split the summation over a into two parts

$$\sum_{a \geq r/p} = \sum_{a \geq 2r/p} + \sum_{r/p \leq a < 2r/p}.$$

According to (5.8) the first sum can be bounded as follows

$$\begin{aligned} \frac{(-1)^k}{r^{-k+v}} \sum_{a \geq 2r/p} \frac{\tau_v(a)\tau_0(pa-r)}{(ap)^k} H_k\left(0, v; \frac{r}{ap}\right) &\ll \\ &\ll \sum_{a \geq 2r/p} (ap)^\epsilon \frac{r^k}{(2ap)^k} \ll \begin{cases} (r/(2p))^k p^\epsilon, & r < p/2 \\ rp^\epsilon/(p2^k), & r > p/2. \end{cases} \end{aligned}$$

Therefore, the contribution of this term doesn't exceed $V_{p,3}(r)$. If $r > p/2$ we estimate the second sum using (5.7) and letting $r = r_0 + sp$, $0 \leq r_0 \leq p-1$. Since $r|l$, $(l, p) = 1$ one has $r_0 \neq 0$. It follows from $a \geq r/p$ that $a \geq s+1$, and, therefore, $ap-r \geq p-r_0$. Thus

$$\begin{aligned} \frac{(-1)^k}{r^{-k+v}} \sum_{r/p \leq a < 2r/p} \frac{\tau_v(a)\tau_0(pa-r)}{(ap)^k} H_k\left(0, v; \frac{r}{ap}\right) &\ll \\ &\ll \frac{(rp)^\epsilon}{k+T} \sum_{r/p \leq a < 2r/p} \frac{\sqrt{r}}{\sqrt{ap-r}} \ll \\ &\ll \frac{(rp)^\epsilon}{k+T} \sqrt{r} \left(\frac{1}{\sqrt{p-r_0}} + \frac{\sqrt{r}}{p} \right) \ll \frac{(rp)^\epsilon}{k+T} \sqrt{r} \left(1 + \frac{\sqrt{r}}{p} \right). \end{aligned}$$

The last summand $W_{p,3}(r)$ can be estimated using corollary 4.5

$$W_{p,3}(r) \ll \sum_{a < r/p} |I(ap/r)| \ll \sum_{a < r/p} \frac{1}{\sqrt{1-ap/r}} \ll \sqrt{r} \left(1 + \frac{\sqrt{r}}{p} \right).$$

Finally,

$$W_p(l) \ll V_p(l) + l^\epsilon \left(1 + \sqrt{l}/p\right) \mathbf{1}_{l>p}.$$

Using (8.6) we obtain the assertion. \square

Lemma 8.6. *One has*

$$(8.8) \quad E_p^3(0, v, \lambda) \ll \frac{\log pT}{\sqrt{dp}}.$$

Proof. Consider

$$(8.9) \quad E_p^3(0, v, \lambda) \ll \frac{1}{\sqrt{dp}} (p^{2v-1} - 1) \log(1 - 2v).$$

This gives

$$E_p^3(0, v, \lambda) \ll \begin{cases} \frac{\log T}{\sqrt{dp}} & v \neq 0, \\ \frac{\log p}{\sqrt{dp}} & v = 0. \end{cases}$$

\square

Lemma 8.7. *One has*

$$(8.10) \quad E_p^4(0, v, \lambda) \ll \frac{(dT k)^\epsilon}{\sqrt{d}}.$$

Proof. First, we estimate the second summand in (7.8). If $v \neq 0$, then it is bounded by $\log T/\sqrt{d}$. Otherwise, letting $v \rightarrow 0$, the poles cancel out and the summand is dominated by $\log kd/\sqrt{d}$. In total, this contributes as $\log kdT/\sqrt{d}$. Now we proceed to estimate the first summand of (7.8) containing the sum of Lerch zeta functions

$$\begin{aligned} \sum_{\substack{b=1 \\ (b, qp)=1}}^{qp} \zeta(0, b/q, 1 + 2v) &= \sum_{n=1}^{\infty} \frac{1}{n^{1+2v}} \sum_{\substack{b=1 \\ (b, qp)=1}}^{qp} \exp\left(\frac{bpn}{qp}\right) = \\ &= \phi(qp) \sum_{n=1}^{\infty} \frac{\mu(q/(n, q))}{n^{1+2v} \phi(q/(n, q))} = \\ &= \phi(qp) \sum_{m|q} \frac{\mu(m)}{\phi(m)} \frac{1}{(q/m)^{1+2v}} \sum_{\substack{n=1 \\ (n, m)=1}}^{\infty} \frac{1}{n^{1+2v}}. \end{aligned}$$

Note that

$$\sum_{\substack{n=1 \\ (n, m)=1}}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} \frac{1}{n^z} \sum_{a|(n, m)} \mu(a) = \zeta(z) \sum_{a|m} \frac{\mu(a)}{a^z}.$$

Thus

$$\sum_{\substack{b=1 \\ (b, qp)=1}}^{qp} \zeta(0, b/q, 1+2v) = \zeta(1+2v) \phi(qp) \sum_{m|q} \frac{\mu(m)}{\phi(m)} \frac{1}{(q/m)^{1+2v}} \sum_{a|m} \frac{\mu(a)}{a^{1+2v}}.$$

Since for $q > 1$

$$\sum_{m|q} \frac{\mu(m)}{\phi(m)} \frac{m}{q} \sum_{a|m} \frac{\mu(a)}{a} = \sum_{m|q} \frac{\mu(m)}{q} = 0,$$

there is no pole at $v = 0$. If $v = it \neq 0$ we use the bound $|\zeta(1+2v)| < \log T$ so that

$$\sum_{\substack{b=1 \\ (b, qp)=1}}^{qp} \zeta(0, b/q, 1+2v) \ll p(Tq)^\epsilon.$$

If $v = 0$

$$\sum_{\substack{b=1 \\ (b, qp)=1}}^{qp} \zeta(0, b/q, 1) = 1/2 \phi(qp) \sum_{m|q} \frac{\mu(m)}{\phi(m)} \frac{m}{q} \sum_{a|m} \frac{\mu(a)}{a} 2 \log \frac{m}{aq},$$

and, therefore,

$$\sum_{\substack{b=1 \\ (b, qp)=1}}^{qp} \zeta(0, b/q, 1) \ll p(Tq)^\epsilon.$$

Combining all the results we have

$$E_p^4(0, v, \lambda) \ll \frac{(dT k)^\epsilon}{\sqrt{d}}.$$

□

Lemma 8.8. *One has*

$$(8.11) \quad E_p^2(0, v, \lambda) \ll \frac{1}{\sqrt{dp}} (Tdp)^\epsilon.$$

Proof. Consider

$$E_p^2(0, v, \lambda) \ll \frac{1}{p} \sum_{q|d} \frac{1}{2\pi\sqrt{d/p}} \left| \sum_{\substack{b=1 \\ (b, qp)=1}}^{qp} \zeta(0, b/qp, 1+2v) \right|.$$

As in the proof of lemma 8.7

$$\sum_{\substack{b=1 \\ (b, qp)=1}}^{qp} \zeta(0, b/qp, 1+2v) = \zeta(1+2v) \phi(qp) \sum_{m|qp} \frac{\mu(m)}{\phi(m)} \left(\frac{m}{qp} \right)^{1+2v} \sum_{a|m} \frac{\mu(a)}{a^{1+2v}}.$$

The pole at $v = 0$ cancels out for $qp > 1$ and

$$\sum_{\substack{b=1 \\ (b, qp)=1}}^{qp} \zeta(0, b/qp, 1 + 2v) \ll (Tqp)^\epsilon.$$

The statement follows. \square

Corollary 8.9. *The contribution of $E_p^j(0, v, \lambda)$, $j = 2, 3, 4$ to (7.10) is bounded by*

$$\begin{aligned} & \frac{2\pi i^{2k}}{p - p^{-1}} \sum_{d|l} \left(\frac{d}{l}\right)^{1/2-v} \times \\ & \times \left(-\frac{E_p^2(0, v, \lambda)}{p^{1/2-v}} - \frac{E_p^3(0, v, \lambda)}{p^{1/2+v}} + \frac{E_p^4(0, v, l)}{p} \right) \ll \frac{(lpTk)^\epsilon}{p^2 \sqrt{l}}. \end{aligned}$$

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